The AdS/CFT Correspondence: Classical, Quantum, and Thermodynamical Aspects

by

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Abstract

Certain aspects of the AdS/CFT correspondence are studied in detail. We investigate the one-loop mass shift to certain two-impurity string states in light-cone string field theory on a plane wave background. We find that there exist logarithmic divergences in the sums over intermediate mode numbers which cancel between the cubic Hamiltonian and quartic "contact term". Analyzing the impurity non-conserving channel we find that leading, non-perturbative terms predicted in the literature are in fact an artifact of these logarithmic divergences and vanish with them. We also argue that generically, every order in intermediate state impurities contributes to the mass shift at leading perturbative order.

The same mass shift is also computed using an improved 3-string vertex proposed by Dobashi and Yoneya. The result is compared with the prediction from non-planar corrections in the BMN limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. It is found to agree at leading order – one-loop in Yang-Mills theory – and is close but not quite in agreement at order two Yang-Mills loops. Furthermore, in addition to the leading non-perturbative power in the 't Hooft coupling, we find that two higher half-integer powers are also miraculously absent. We extend the analysis to include discrete light-cone quantization, considering states with up to three units of p^+.

We study the weakly coupled plane-wave matrix model at finite temperature. This theory has a density of states which grows exponentially at high energy, implying that the model has a phase transition. The transition appears to be of first order. However, its exact nature is sensitive to interactions. We analyze the effect of interactions by computing the relevant parts of the effective potential for the Polyakov loop operator to three loop order. We show that the phase transition is indeed of first order. We also compute the correction to the Hagedorn temperature to two loop order.

Finally, correlation functions of 1/4 BPS Wilson loops with the infinite family of 1/2 BPS chiral primary operators are computed in $\mathcal{N} = 4$ super Yang-Mills theory by summing planar ladder diagrams. Leading loop corrections to the sum are shown to vanish. The correlation functions are also computed in the strong-coupling limit by examining the supergravity dual of the loop-loop correlator. The strong coupling result is found to agree with the extrapolation of the planar ladders. The result is related to known correlators of 1/2 BPS Wilson loops and 1/2 BPS chiral primaries by a simple re-scaling of the coupling constant, similar to an observation made in the literature, for the case of the 1/4 BPS loop vacuum expectation value.

Contents

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Foreword

This thesis collects the work of four publications by the author concerning quantum, classical, and thermodynamical aspects of the AdS/CFT correspondence. The thesis begins with an introductory chapter which provides the reader with the necessary background in non-abelian gauge theory and the 't Hooft expansion, the non-renormalizable nature of point-particle quantum gravity, supersymmetry, modern string theory, and the AdS/CFT correspondence itself. The main matter of the thesis begins with chapter 2 where the reader is introduced to the plane-wave version of the AdS/CFT correspondence and to light-cone string field theory in that context. The original work of the author published in [89] is presented in section 2.3, while that of [90] is presented in section 2.4. These works concern divergence cancellations in string loop corrections and the comparison of those corrections with their gauge theory duals. Chapter 3 begins with an introduction to the matrix model of M-theory, and specifically to the plane-wave matrix model. The original work of the author [120] concerning the deconfinement phase transition found in this model is presented in section 3.3. Chapter 4 begins with a brief introduction to the Wilson loop in the AdS/CFT correspondence. In section 4.2, original work of the author [135] concerning the two point functions of chiral primary operators with a certain 1/4 BPS circular Wilson loop is presented.

Chapter 1

Introduction

> His tongue, continuous before and apt
> For utterance, severs; and the other's fork
> Closing unites. That done, the smoke was laid.
> The soul, transform'd into the brute, glides off,
> Hissing along the vale, and after him
> The other talking sputters;
>
> — Dante's *Inferno*, Canto 25

It is a well known observation that in any endeavour, the tension of apparent contradictions leads to a higher understanding - one which naturally fuses those into a whole greater than the sum of its parts. Such a tension exists in theoretical physics, between the description of the strongest force in nature, and the weakest. String theory in general and the AdS/CFT correspondence in particular, are emerging as a fusion of the understanding of these two forces; a symbiosis with the potential to answer questions beyond the scope of either and to probe the very structure of space and time themselves. In order to understand this correspondence, we must know something about these two forces, and their individual descriptions.

1.1 Strong nuclear force

The strong nuclear force is responsible for the cohesion of matter at the smallest known scales - inside the particles which compose the nuclei of atoms - a scale of 10^{-15} m. The modern description of this force is known as Quantum Chromodynamics or QCD. QCD is a non-abelian gauge theory described by the Yang-Mills action

$$S = -\frac{1}{4}\int d^4x \ \mathrm{Tr}\ F_{\mu\nu} F^{\mu\nu} + S_m, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\, g_{YM}\, [A_\mu, A_\nu] \tag{1.1}$$

where the $A_\mu(x)$ are matrix-valued four-vectors in the adjoint representation of $SU(3)$, and the coupling constant in the theory is g_{YM}. The action for the matter content of the theory - the quarks - has been indicated by S_m. As is usual in quantum field theories, this bare coupling is renormalized, and the physical strength of the force described by the theory is given by the renormalized coupling $\tilde{g}_{YM}(k)$ which is a function of the energy scale k of the process being described. A remarkable feature of this renormalization earned Politzer, Gross, and Wilczek the Nobel prize in physics 2004. The coupling g_{YM} of QCD, unlike other quantum field theories, decreases with increasing energy scale k, so that at high energies, the theory becomes free. This property therefore earned the name *asymptotic freedom*. We see here that the strongest force in nature, is actually weak if the relevant energy scales are high enough.

Often in quantum field theory our only analytic tool is perturbation theory. The same is true for QCD. If we would like to calculate the expectation value of an observable $\mathcal{O}$, we need to evaluate the path-integral[1]

$$\langle \mathcal{O} \rangle = \frac{\int [dA_\mu]\, \mathcal{O}\, e^{iS}}{\int [dA_\mu] e^{iS}}. \tag{1.2}$$

This is accomplished by Taylor-expanding the exponential e^{iS}, out to the desired order of accuracy. This procedure is only sensible when g_{YM} is small. For QCD, this procedure then only works for very high-energy processes. Indeed this is the regime where QCD has been tested in particle accelerators, and has successfully described the dynamics witnessed there. But at terrestrial energy scales, like those found roughly anywhere cooler than inside the sun, $g_{YM} \sim 1$, and perturbation theory is useless. Now we see that there are two issues, one is that the strong force is only sometimes strong, and the second is that we can only use our quantum field theory to (analytically) describe its nature when it is weak. Lattice field theory is a numerical technique which allows strong-coupling answers to be squeezed out of (1.1) and has been successful in describing some aspects of those dynamics. However, an analytical technique remains out of reach, and greatly desired.

1.2 Gravity and renormalization

Gravity is the force responsible for structure at the largest known scales - those of the known universe - some 10^{25} m in size. The gravitational force is 40 orders of magnitude weaker than the strong force. The modern description of gravity is entirely classical, it says nothing about $\hbar$, the scale at which quantum fluctuations become important. In this respect it is radically different from QCD, for which *only* a quantum description is sensible, due to its fantastically short range. Gravity's modern description was given birth to by Einstein, who successfully unified the force with the precepts of special relativity - that is Lorentz invariance. It is captured by the Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R \qquad R = g^{\mu\nu} R_{\mu\nu} \qquad R_{\mu\nu} = \partial_\mu \Gamma^\rho_{\rho\nu} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu} - \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} \tag{1.3}$$

where R is the Ricci scalar built out of $g_{\mu\nu}$, the metric of space-time, and G is the Newton constant, or universal constant of gravitation. Already at this level we note similarities between the descriptions of these vastly divergent forces. The Christoffel connection $\Gamma^\rho_{\mu\nu}$ is analogous to the gauge field A_μ of (1.1) and the Ricci tensor $R_{\mu\nu}$ is a sort of "field-strength" of $\Gamma^\rho_{\mu\nu}$ in the same sense that $F_{\mu\nu}$ is of A_μ. We see immediately that the two theories are non-linear (non-abelian), and so share the characteristic that their fields are sources for

[1] We are being schematic here, in an attempt to maintain clarity. A more precise statement is that $\langle \Omega | \mathcal{T}\, \mathcal{O}\, | \Omega \rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\int [dA_\mu][d\,\mathrm{matter}]\mathcal{O} \exp\left[i \int_{-T}^{T} d^4x \mathcal{L}\right]}{\int [dA_\mu][d\,\mathrm{matter}] \exp\left[i \int_{-T}^{T} d^4x \mathcal{L}\right]}$, where $S = \int d^4x\, \mathcal{L}$, $|\Omega\rangle$ is the ground state of the interacting theory, and $\mathcal{T}$ indicates time-ordering, c.f. [28].

themselves. However, early attempts to push this analogy further by quantizing gravity met with failure.

It is simple to see that there is a scale at which one expects gravity to be modified by quantum mechanics. Take for example a black hole formed by a very heavy particle. When the Compton wavelength of the particle is comparable to the Schwarzschild radius

$$\frac{\hbar}{m\,c} \sim \frac{2\,G\,m}{c^2} \tag{1.4}$$

we expect that classical gravity ought to be invalid. This occurs for $m = M_{pl} \simeq 10^{19}$ GeV, or for length scales $l_{pl} \simeq 10^{-35}$ m. At these unimaginably high energies, an accurate description of gravity would naïvely be given by a quantization of the classical theory into a quantum field theory of gravity. It is, however, precisely that characteristic of such theories which is responsible for asymptotic freedom in QCD, which cripples such an attempt at the first step.

The renormalization of the coupling constant in a quantum field theory, such as QCD, arises in the treatment of integrals over the momenta of intermediate virtual particles. These integrals formally diverge, but may be made finite by placing an upper-bound on the momenta being integrated over. This procedure is very sound physically, because one expects the quantum field theory at hand to be an *effective* theory, valid at the scale in question, but eventually superseded at some higher energy, where new physics is expected to be active. In condensed matter physics, this idea was understood early on, because the cut-off is the very physical scale of the atomic size. Once cut off, the integrals in QCD produce pieces proportional to the cut-off but independent of the energy scale of the process being described, and other pieces independent of the cut-off, but dependent on the relevant energy scale. The cut-off dependent pieces are interpreted in much the same way that an absolute potential energy is - it is irrelevant - only potential differences are physical. It is then the cut-off independent, energy scale dependent or *running* quantities which correspond to physical attributes of the theory.

If the coupling constant in a quantum field theory is dimensionless, then probability amplitudes may be expressed as polynomials in it. This is the case for QCD. Should the coupling constant g have negative mass-dimension $-p$, then probability amplitudes can only be described as polynomials in the dimensionless combination $\Lambda^p\,g$, where Λ is the momentum cut-off. This is a non-renormalizable quantum field theory, whose cut off momentum integrals do not contain pieces independent of the cut-off scale. Because we cannot - regardless of the energy scale of the process being described - arrive at a prediction independent of the cut-off scale, and since we don't know with any precision what this scale is, we cannot make any definite predictions with such a theory. As can be seen from (1.4), the coupling constant in gravity G is proportional to m^{-2} if we set $\hbar = c = 1$. Thus gravity has a coupling constant with negative mass dimension and so is a non-renormalizable quantum field theory.

It is this fact that set the strongest and weakest forces in the universe at loggerheads. Indeed, it set gravity apart from all three of the other fundamental forces, which were successfully described by an aggregate, renormalizable quantum field theory called the *standard model* by the 1970's.

1.3 Early string theory and large N

String theory was born in an attempt to describe the strong nuclear force before the days of QCD. One of the early observations was that there was a zoo of mesons, whose masses m were related to their spins via $J = \alpha' m^2$, where α' is a constant known as the Regge slope. It was soon realized that a quantum string gave rise to such a relation. With the benefit of hindsight, we can see how this stringy-ness is manifested in mesons. We now know that a meson is a quark-antiquark bound state, whose colour field lines are confined into a *flux tube* as shown in figure 1.1. It is this flux tube which behaves as a string of a given tension. An

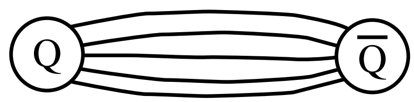


Figure 1.1: The field lines between two quarks in the low energy regime of QCD.

empirical formula for meson scattering was put forward by Veneziano [1], which was later shown to be derivable from string theory. However in the late 60's experimental data began to show that the Veneziano amplitude gave an incorrect large energy behaviour, and soon after QCD was adopted as the correct description of the strong force.

This turn of events still left the question of how the action (1.1) could possibly encode string-like dynamics in the strong coupling regime. In 1974 't Hooft [2] made a remarkable leap forward in this direction. String perturbation theory naturally organizes itself into a *genus expansion*, see figure 1.2. Whereas in a regular quantum field theory each vertex

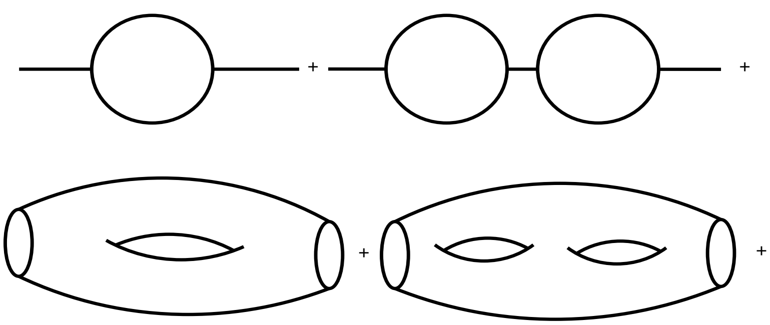


Figure 1.2: Comparison of quantum field theory perturbative expansion to that of string theory. A closed string sweeps-out a two dimensional *worldsheet* whose genus represents the number of powers of the string coupling constant associated with the process.

would contribute a power (or two) of the coupling constant, the same rôle in string theory is played by the genus of the worldsheet. 't Hooft discovered that such a genus expansion lay hidden in the theory described by (1.1). In order to see this we take the gauge group of the theory to be $SU(N)$, where we will eventually want to consider N large. We write the gauge fields as $A_\mu = g_{YM}\, A^a_\mu\, t^a$, where t^a are the generators of $SU(N)$ with $a = 1, \ldots, N$, and we have rescaled the fields by the coupling constant. This allows us to write the action in the following form[2]

$$S = -\frac{1}{2\, g_{YM}^2} \int d^4x \,\mathrm{Tr}\, F_{\mu\nu}\, F^{\mu\nu} \tag{1.5}$$

where now $F_{\mu\nu}$ contains no factors of g_{YM}. We may express the gauge degrees of freedom as

$$A^a_\mu t^a_{\bar{i}j} \rightarrow (A_\mu)_{\bar{i}j} \tag{1.6}$$

where $\bar{i}, j$ is an (anti) fundamental index running from $1, \ldots, N$. In this language the propagators have the following index structure

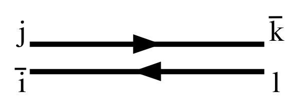
$$\Big\langle\, (A_\mu(x))_{\bar{i}j}\, (A_\nu(y))_{\bar{k}l} \,\Big\rangle \sim \big(\delta_{\bar{i}l}\delta_{\bar{k}j} - \frac{1}{N}\delta_{\bar{i}j}\delta_{\bar{k}l}\big) \tag{1.7}$$

and therefore to leading order in the large N limit, the second term in the gauge field propagator may be ignored. In fact, this second term disappears entirely if one considers the gauge group $U(N)$ instead of $SU(N)$, and then everything that follows here is exactly (instead of approximately) true. In non-abelian gauge theory, the gauge field $A^a_\mu(x)$ always transforms in the adjoint representation of the gauge group. The interpretation of the picture which emerges here is that an adjoint field $\phi^a(x)$ may be represented as a direct product of fundamental and anti-fundamental fields, $\phi_{\bar{i}}(x)\phi_j(x) = \phi_{\bar{i}j}(x)$. In group theory language this is the statement

$$\bar{N} \otimes N = \text{adjoint} \oplus \text{singlet} \tag{1.8}$$

but for large N the singlet contribution is suppressed, as per (1.7). Thus the adjoint gauge fields are in a sense quark/anti-quark composites, stressing the *flux tube* interpretation. 't Hooft developed a diagram notation based on this fact, called the *fat graph* notation,

$$\Big\langle\, (A_\mu(x))_{\bar{i}j}\, (A_\nu(y))_{\bar{k}l} \,\Big\rangle \qquad \sim$$



in this notation the three and four point vertices are given by the diagrams shown in figure 1.3.

We now introduce the quantity λ which is known as the 't Hooft coupling, $\lambda = g_{YM}^2 N$. Glancing back at (1.5), it can be seen that the three and four point vertices come with a factor of $1/g_{YM}^2 = N/\lambda$ whereas the gauge field propagator is proportional to $g_{YM}^2 = \lambda/N$. Also, in a given diagram, when an arrowed line closes on itself (forms a loop) it supplies a factor of $\delta_{ii} = N$. Consider now the set of all vacuum diagrams, an example of which is shown in figure 1.4. A diagram with V vertices, E propagators, and F loops will therefore be proportional to

$$N^{V-E+F}\lambda^{E-V} \tag{1.9}$$

[2]We have left the matter action S_m out of this discussion, for simplicity.

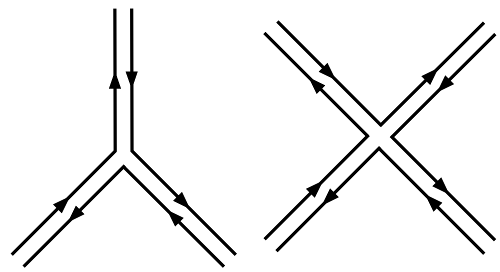


Figure 1.3: Vertices in the ‘t Hooft model.

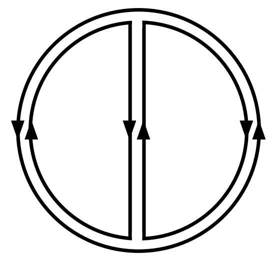


Figure 1.4: A vacuum diagram of the ’t Hooft model.

We have chosen the letters V and E because if we collapse the double lines to single ones, then the diagram has V vertices, and E edges, as in figure 1.5. The letter F is chosen because each loop forms a *face* in the double-line diagram. The combination $\chi = V - E + F$ is recognized

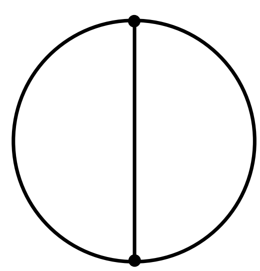


Figure 1.5: The vertices and edges of the diagram in figure 1.4.

as the Euler character, which implies a connection between the diagrams and surfaces of a given genus g, since $\chi = 2 - 2g$. Thus we have that a given diagram is proportional to

$$N^{2-2g}\lambda^{E-V} \tag{1.10}$$

and so diagrams corresponding to surfaces of higher genus are suppressed by successive powers of $1/N^2$. An example of a higher genus diagram is shown in figure 1.6.

Referring to figure 1.4, we see that $V = 2$, $E = 3$, and $F = 3$, and therefore the power of N associated with this diagram is $N^{2-3+3} = N^2$ or equivalently the genus g is 0. The genus

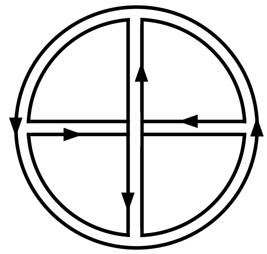


Figure 1.6: A diagram associated with a surface of genus $g = 1$.

0 graphs are given a special name, they are called *planar* graphs. This is because they can be drawn on a plane. In contrast figure 1.6 shows a graph with $V = 4$, $E = 6$, and $F = 2$, and thus is proportional to N^0 or is genus 1 and obviously can not be drawn on a plane due to the crossing central propagators; a handle would need to be added to the plane in order to draw this graph. The sum of all the vacuum diagrams takes the following form

$$\sum_g N^{2-2g} \mathcal{F}_g(\lambda) \tag{1.11}$$

$$\mathcal{F}_g(\lambda) = \sum_n C_n^g \lambda^n \tag{1.12}$$

where the function $\mathcal{F}_g(\lambda)$ is the sum of all diagrams of genus g, which is naturally a power series in the 't Hooft coupling. For example $\mathcal{F}_0(\lambda)$ would be the sum of all planar diagrams. The point of interest here is that in the large N limit, we have a perturbative genus expansion. In string theory, the very same type of expansion arises in the calculation of amplitudes. It is this connection to string theory which makes the 't Hooft large N expansion an important observation; it connects QCD-type quantum field theory to string theory.

1.4 Modern string theory

After losing the bid to describe the strong nuclear force, string theory was revived in the 1980's when it was realized that it could be used as a candidate for a quantum theory of gravity and perhaps even a grand unified theory of physics. The divergences which plagued the quantum field theory approach to quantizing gravity disappear with string theory. This can be traced to the delocalization of the interaction vertices enjoyed by stringy Feynman diagrams, as shown in figure 1.2. We will review string theory from the perspective of the Green-Schwarz superstring, as this will be most relevant for the work on light-cone string field theory on the plane-wave presented in chapter 2. We will not reference very widely in this section; the relevant references are to be found in the standard textbooks such as [35][36].

1.4.1 Preliminaries

The dynamics of a string are defined by the requirement that at the classical level, they lead to a minimization of the proper area swept out by the string's worldsheet as it propagates through a target spacetime $G_{\mu\nu}$, see figure 1.7. The string worldsheet is embedded into the

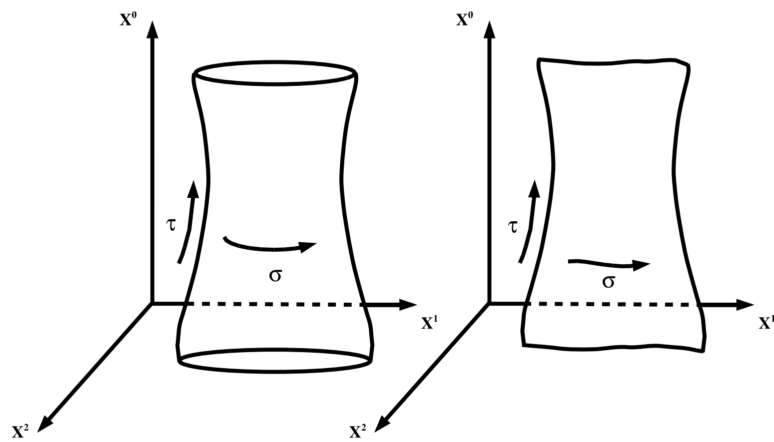


Figure 1.7: A closed (left) or open (right) string worldsheet is described by embedding functions $X^\mu(\sigma, \tau)$.

target spacetime by d embedding functions $X^\mu(\sigma, \tau)$, where (σ, τ) are the coordinates on the worldsheet and d is the dimension of the target spacetime. The Polyakov action for the string is given by

$$S_P = -\frac{T}{2} \int d\sigma \int d\tau \sqrt{|\det h|}\, h^{ab}\, \partial_a X^\mu\, G^{\mu\nu}(X)\, \partial_b X^\nu \tag{1.13}$$

where a and b take on values 0 or 1 corresponding to the coordinates τ and σ respectively, and $h_{ab}(\sigma, \tau)$ is known as the worldsheet metric. The energy per unit length of the string, or *string tension*, is given by T. The Polyakov action respects the symmetries of the target spacetime, but also respects two further symmetries: Weyl (or conformal) invariance, and reparametrization invariance. Weyl invariance is simple to see. Consider rescaling the worldsheet metric as follows $h'_{ab} = \omega(\sigma, \tau)\, h_{ab}$. Only for two-dimensional metrics will the combination $\sqrt{|\det h|}\, h^{ab}$ be invariant. This is a very powerful symmetry in string theory. It tells us that the worldsheet theory is conformally invariant. Reparametrization invariance is the statement that we may paint on to the worldsheet any coordinates we see fit; the dynamics can not depend on the coordinate system chosen. This symmetry may be expressed as follows

$$\sigma \to \sigma'(\sigma, \tau) \qquad \tau \to \tau'(\sigma, \tau). \tag{1.14}$$

We thus have three free functions with which to gauge-fix the worldsheet metric h_{ab} : two reparametrization and one Weyl re-scaling. However, being a symmetric 2×2 matrix, h_{ab}

has only three degrees of freedom. We are therefore free to set it to the Minkowski metric diag$(-1, 1)$.

In order to analyze the equations of motion for the fields h_{ab} and X^μ, we will temporarily set the target space to flat d-dimensional Minkowski space. The equations of motion are then

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^\mu(\sigma, \tau) = 0 \qquad \frac{\delta S_P}{\delta h_{ab}} = 0 \tag{1.15}$$

and so we have the free two-dimensional wave equation governing the embedding functions, while the equation of motion for h_{ab} may be restated using the energy-momentum tensor for the worldsheet theory

$$T_{ab} \equiv -\frac{2}{T} \frac{1}{\sqrt{|\det h}} \frac{\delta S_P}{\delta h_{ab}} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\, h_{ab}\, h^{cd} \partial_c X^\mu \partial_d X_\mu = 0. \tag{1.16}$$

Once the Minkowski gauge has been chosen for the worldsheet metric, (1.16) must be imposed as a constraint - this is known as the *Virasoro constraint*. The solutions to the wave equation for the embedding functions come in two topologies, which are differentiated by a choice of boundary condition. We may impose one of

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau) \qquad \text{or} \qquad \partial_\sigma X^\mu(0, \tau) = \partial_\sigma X^\mu(\pi, \tau) = 0 \tag{1.17}$$

where we have taken the range of σ to be $[0, \pi]$. The first of these describes closed strings, and the second open strings.

1.4.2 Mode expansions and light-cone gauge

The solution of (1.15) for closed strings is as follows,

$$\begin{aligned} X^\mu(\sigma, \tau) &= X^\mu_L + X^\mu_R \qquad \text{where} \\ X^\mu_L &= \frac{1}{2} x^\mu_L + \alpha' p^\mu_L (\tau + \sigma) + \frac{i}{\sqrt{2\alpha'}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}^\mu_n\, e^{-2in(\tau+\sigma)} \\ X^\mu_R &= \frac{1}{2} x^\mu_R + \alpha' p^\mu_R (\tau - \sigma) + \frac{i}{\sqrt{2\alpha'}} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n\, e^{-2in(\tau-\sigma)} \end{aligned} \tag{1.18}$$

where we have introduced $\alpha' \equiv (2\pi T)^{-1}$, which is the Regge slope. The quantities $\frac{1}{2}(x^\mu_L + x^\mu_R)$ and $\frac{1}{2}(p^\mu_L + p^\mu_R)$ are the center of mass coordinates and momenta[3], respectively. Since the string is closed, we must take $p^\mu_L = p^\mu_R$ in a topologically trivial target space[4]. The α^μ_n and $\widetilde{\alpha}^\mu_n$ are the amplitudes of the n-th left-moving and right-moving vibration modes, respectively. Reality of X^μ enforces $(\alpha^\mu_n)^* = \alpha^\mu_{-n}$, and $(\widetilde{\alpha}^\mu_n)^* = \widetilde{\alpha}^\mu_{-n}$. The Virasoro constraint (1.16) is

$$\dot{X} \cdot X' = 0 = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) \tag{1.19}$$

[3]Note that momentum is defined as $\int_0^\pi d\sigma \delta L/\delta \dot{X}^\mu = T \int_0^\pi d\sigma \dot{X}^\mu(\sigma, 0)$.

[4]An open string must obey the Neumann boundary condition $\partial_\sigma X^\mu(\sigma)|_{\sigma=0,\pi} = 0$. This not only sets $p^\mu_L = p^\mu_R$ but also enforces $\widetilde{\alpha}^\mu_n = \alpha^\mu_n$.

where we use the prime to denote differentiation by σ and the dot for differentiation by τ. Further the dot product refers to contraction of Lorentz indices.

There are different methods of quantizing the string, but we will concentrate on light-cone quantization. This method is attractive because it eliminates unphysical degrees of freedom at the outset, so that every quantum state is physical, and there is no need to worry about ghosts. The drawback of the method is that Lorentz invariance becomes obscured and is no longer manifest. To begin, we note that fixing h_{ab} to the Minkowski metric has not completely used up the gauge freedom. Indeed, under a reparametrization $\xi^0 = \partial\tau/\partial\tau'$, $\xi^1 = \partial\sigma/\partial\sigma'$, h_{ab} transforms as follows

$$\delta h^{ab} = \xi^c \partial_c h^{ab} - \partial_c \xi^a h^{cb} - \partial_c \xi^b h^{ac}. \tag{1.20}$$

If we accompany this by a Weyl rescaling $(1 + \omega(\sigma, \tau))$ such that

$$\partial^a \xi^b + \partial^b \xi^a = \omega \eta^{ab} \tag{1.21}$$

where η^{ab} is the Minkowski metric, then this combination leaves the choice $h^{ab} = \eta^{ab}$ invariant. Consider the following coordinates $\sigma^{\pm} = \tau \pm \sigma$. In these coordinates (1.21) for $a \neq b$ becomes

$$\partial_+ \xi^- = 0 \qquad \partial_- \xi^+ = 0. \tag{1.22}$$

This implies that we are free to change σ^+ by any function of σ^+: $\sigma^+ \to \tilde{\sigma}^+(\sigma^+)$, and similarly $\sigma^- \to \tilde{\sigma}^-(\sigma^-)$. This is a powerful residual gauge symmetry which allows for light-cone gauge quantization.

The manifest Lorentz invariance of the target space is broken in the light-cone gauge, by singling-out two directions to be the so-called light-cone directions

$$X^{\pm} = \frac{1}{\sqrt{2}} \left(X^0 \pm X^{d-1} \right) \qquad X^{\mu} = (X^-, X^+, X^i) \tag{1.23}$$

where $i = 1, \ldots, d-2$. In the previous paragraph, we saw that we are free to reparametrize the worldsheet coordinates. To this end, and in light of (1.18) we choose

$$\sigma^+ = \frac{1}{\alpha' p^+} \left(X_L^+ - \frac{1}{2} x^+ \right) \qquad \sigma^- = \frac{1}{\alpha' p^+} \left(X_R^+ - \frac{1}{2} x^+ \right). \tag{1.24}$$

Now we have a rather "natural" embedding where the worldsheet time τ is simply given by the light-cone + direction, since,

$$X^+(\sigma, \tau) = X_L^+ + X_R^+ = x^+ + 2\alpha' p^+ \tau. \tag{1.25}$$

We still need to impose the Virasoro constraints (1.19), which we may write equivalently as $(\dot{X} \pm X')^2 = 0$. In light-cone coordinates, this is

$$(\dot{X}^- \pm X^{-\prime}) = \frac{1}{4\alpha' p^+} (\dot{X}^i \pm X^{i\prime})^2 \tag{1.26}$$

where we have used (1.25). From this expression $X^-(\sigma, \tau)$ is completely fixed in terms of the $X^i(\sigma, \tau)$. What we see here is that, in fact, there are only $d-2$ physical vibratory degrees of freedom; these are the $X^i(\sigma,\tau)$. The $X^\pm$ are non-dynamical, and fixed by gauge freedom and the imposition of constraints. The physical idea here is that longitudinal modes are non-physical and do not correspond to string dynamics. The transverse oscillations captured by the mode expansions of the $X^i(\sigma,\tau)$, along with any center of mass motion, completely capture the string's dynamics.

The Virasoro constraints are enforced through the Fourier components of the stress-energy tensor (1.16). In light-cone gauge, we have

$$\begin{aligned} L_m &= \frac{T}{2}\int_0^\pi d\sigma\, e^{-2im\sigma}\, T_{--} = \alpha' \frac{p_R^2}{8} + \frac{1}{2}\sum_{n\neq 0} \alpha_{m-n}\cdot\alpha_n \\ \widetilde{L}_m &= \frac{T}{2}\int_0^\pi d\sigma\, e^{-2im\sigma}\, T_{++} = \alpha' \frac{p_L^2}{8} + \frac{1}{2}\sum_{n\neq 0} \widetilde{\alpha}_{m-n}\cdot\widetilde{\alpha}_n \end{aligned} \tag{1.27}$$

and must have that $L_m = \widetilde{L}_m = 0$ for all m. The condition $L_0 + \widetilde{L}_0 = 0$ then gives us[5]

$$(\text{mass})^2 = -p^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \widetilde{\alpha}_{-n}\cdot\widetilde{\alpha}_n) \tag{1.28}$$

where we have used the fact that $p_L^\mu = p_R^\mu = p^\mu$ for closed strings. This is a truly beautiful result, for it tells us that the internal excitations of the string worldsheet are reflected as spacetime mass in the target space; an excited string is heavy. Another important constraint arising from L_0 and $\widetilde{L}_0$ is the *level matching condition*. This stems from the condition $L_0 - \widetilde{L}_0 = 0$. This tells us that

$$\sum_{n\neq 0}\alpha_{-n}\cdot\alpha_n = \sum_{n\neq 0}\widetilde{\alpha}_{-n}\cdot\widetilde{\alpha}_n \tag{1.29}$$

or, in other words, the degree of excitation of left moving modes must be matched by that of the right-moving modes. As we will see in a later section, the level-matching condition is modified when the target space contains topologically non-trivial cycles.

1.4.3 Supersymmetry: Why?

Supersymmetry is an enlargement of the symmetry group of spacetime obtained by a grading of that algebra. In terms of particles and fields propagating in spacetime, it is more simply understood as the statement that there is a symmetry relating fermionic and bosonic physical degrees of freedom such that, for every bosonic state of mass m, there exists a fermionic *superpartner* of the same mass, and the same quantum numbers generally, with the obvious exception of spin. Supersymmetry has grown to be an attractive concept in theoretical physics. Pessimistically, one might say this is because an enlargement of symmetry allows for an enlargement of calculational techniques, or at least an enlargement of ease in developing calculational techniques. Optimistically, supersymmetry does go a certain distance towards

[5] When the string is quantized, the oscillators are promoted to operators and a normal ordering constant modifies this relation so that $L_0 + \widetilde{L}_0 - 2a$, instead, annihilates a physical state.

solving the cosmological constant problem, the hierarchy problem, and when applied to the standard model, predicts a unification of the strong, weak, and electromagnetic coupling constants, see figure 1.8 for a cartoon of this result. This last point peaks interest in so far as the possible indication that at some high energy scale, a grand unified supersymmetric theory may exist, which flows down to our standard model at low energies.

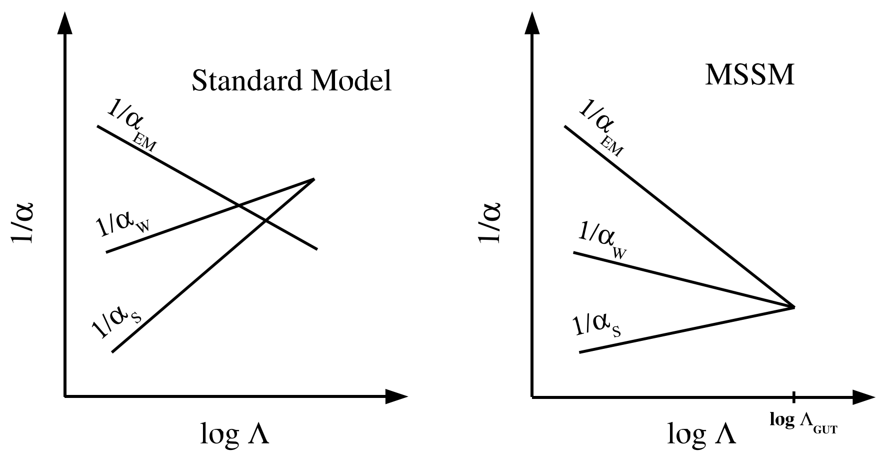
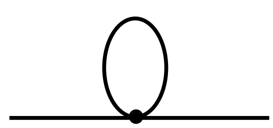


Figure 1.8: Under the minimal supersymmetric extension of the standard model (MSSM), the extrapolated couplings for the strong, weak, and electromagnetic fields are equal at an energy scale Λ_{GUT}, the *grand unified theory* scale.

The cosmological constant problem is an apparent mismatch between the expected vacuum energy of the standard model, and the observed value in nature inferred via cosmology. The discrepancy is an embarrassing 120 orders of magnitude. The vacuum energy in the standard model is formally infinite as it corresponds to the sum of the zero-point energies of all the modes of all the fields. This infinity is cut off conservatively at the Planck scale 10^{19} GeV, since we expect the standard model to lose its validity at least by this energy. A standard model with unbroken supersymmetry would actually give zero for the vacuum energy. This is a general statement about supersymmetric theories: the ground state energy is always identically zero. Of course the cosmological constant observed in nature is not zero, but there are methods available to softly break supersymmetry leading to a greatly reduced, non-zero, vacuum energy.

The hierarchy problem concerns the mass m of the Higgs boson. The renormalization of this mass is controlled by the quadratic divergence encountered in the quantum corrections to its propagator



$$\rightarrow \qquad \delta m^2 \propto M^2$$

where M is the mass of the particle in the loop. Therefore if there are heavy particles in a theory, they will cause the renormalized Higgs mass to be too large. For example, string theory will give Planck mass particles, thus causing the Higgs mass and hence the electroweak scale to be Planck scale. In fact the electroweak scale is about 100 GeV. In a supersymmetric theory, the corrections would also include a fermion loop, which would cancel out the mass shift. This cancellation would persist at higher order loops in perturbation theory effectively protecting the Higgs mass against quantum corrections. Without such a mechanism, the only recourse is to "fine-tune" the bare (unrenormalized) Higgs mass so as to end-up with the observed value after renormalization. This fine-tuning is viewed as an extremely unnatural procedure in a fundamental theory of physics, and is generally unpalatable to most researchers. Supersymmetry offers a more universal resolution of this issue.

Obviously, supersymmetry is not an exact symmetry of nature at currently probed energy scales; we do not see superpartners of the elementary particles. This is generally taken to mean that supersymmetry is a broken symmetry, and that there is a scale which sets when this breaking occurs, and which determines the masses of the superpartners. Particle experimentalists are ever pushing up the energy of their accelerators in hopes that, amongst other things, the superpartner masses will cross into view.

1.4.4 Supersymmetry: Details and implementation on the string

The Poincaré group is a realization of the symmetries manifest in flat Minkowski spacetime - translations, rotations, and boosts. The generators of these transformations are given by P_μ, J_i, and K_i respectively and obey an algebra given by

$$\begin{aligned} [J_i, J_j] = i\epsilon_{ijk} J_k \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k \qquad [J_i, K_j] = i\epsilon_{ijk} K_k \\ [J_i, P_j] = i\epsilon_{ijk} P_k \qquad [J_i, P_0] = 0 \qquad [K_i, P_j] = -i\delta_{ij} P_0 \qquad [K_i, P_0] = -iP_i \end{aligned} \tag{1.30}$$

where $\mu = 0, \ldots, d-1$ is a spacetime index, while $i, j, k = 1, \ldots, d-1$ are spacial indices. This is more compactly expressed in terms of the Lorentz generators $M_{\mu\nu} = -M_{\nu\mu}$ defined as $M_{0i} = K_i$ and $M_{ij} = \epsilon_{ijk} J_k$

$$\begin{aligned} [P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu} P_\nu + i\eta_{\rho\nu} P_\mu \\ [M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\nu\rho} M_{\mu\sigma} - i\eta_{\mu\rho} M_{\nu\sigma} - i\eta_{\nu\sigma} M_{\mu\rho} + i\eta_{\mu\sigma} M_{\nu\rho}. \end{aligned} \tag{1.31}$$

where $\eta_{\mu\nu} = \mathrm{diag}(+, -, \ldots, -)$. Supersymmetry enlarges this group via the introduction of spinorial generators Q^I_α and $\bar{Q}^I_{\dot\alpha}$, where α and $\dot\alpha$ are spinor indices, while $I = 1, \ldots, \mathcal{N}$ labels the number of supersymmetries. The enlargement of (1.31) is as follows

$$\begin{aligned} [P_\mu, Q^I_\alpha] = 0 \qquad [P_\mu, \bar{Q}^I_{\dot\alpha}] = 0 \qquad [M_{\mu\nu}, Q^I_\alpha] = i\left(\Gamma_{\mu\nu}\right)_\alpha^{\ \beta} Q^I_\beta \qquad [M_{\mu\nu}, \bar{Q}^I_{\dot\alpha}] = i\left(\bar{\Gamma}_{\mu\nu}\right)_{\dot\alpha}^{\ \dot\beta} \bar{Q}^I_{\dot\beta} \\ \{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\Gamma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ} \qquad \{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\left(Z^{IJ}\right)^* \end{aligned} \tag{1.32}$$

where some definitions are in order. The bar is defined as follows: $\bar{Q} = Q^\dagger \Gamma^0$. The Γ^μ are a representation of the d-dimensional Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{1.33}$$

and $\Gamma^{\mu\nu} \equiv -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$. The Z^{IJ} are an $\mathcal{N} \times \mathcal{N}$ matrix of *central charges* which is necessarily antisymmetric in I, J and therefore exists only for $\mathcal{N} > 1$. This is called *extended* supersymmetry.

There are two main methods of implementing supersymmetry on the string worldsheet. They are referred to as the Neveu-Schwarz-Ramond or NSR string, and the Green-Schwarz string. The latter makes the resulting supersymmetry in the target spacetime more explicit, and will also be more relevant for describing superstrings in the plane-wave background of chapter 2. For these reasons, the Green-Schwarz formalism is developed here. It is most instructive to consider the supersymmetrized action of a point particle, rather than a string, first. A massless point particle in Minkowski spacetime has the following action

$$S = \int d\tau \frac{1}{h(\tau)} \dot{x}^\mu(\tau)\dot{x}_\mu(\tau) \tag{1.34}$$

where h is a worldline metric relating an interval in τ to a physical time interval. The embedding function $x^\mu(\tau)$ describes the worldline of the particle through spacetime. We can render this action supersymmetric through the introduction of some fermionic partners for the bosonic fields x^μ. These we denote $\theta^A_a(\tau)$ where $A = 1, \ldots, \mathcal{N}$ and a is a spacetime spinor index which will be suppressed in what follows. It can be verified that the following generalization of (1.34)

$$S = \int d\tau \frac{1}{h(\tau)} \left(\dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A\right)^2 \tag{1.35}$$

is invariant under the supersymmetry variations

$$\delta\theta^A = \epsilon^A \qquad \delta x^\mu = i\bar{\epsilon}^A\Gamma^\mu\theta^A \qquad \delta\bar{\theta}^A = \bar{\epsilon}^A \qquad \delta h = 0 \tag{1.36}$$

where $\epsilon, \bar{\epsilon}$ are spinors independent of τ. Thus there are $\mathcal{N}$ supersymmetries obeyed by this action. The equations of motion of the fields in (1.35) are given by

$$p^2 = 0 \qquad \dot{p}^\mu = 0 \qquad \Gamma \cdot p\,\dot{\theta}^A = 0 \tag{1.37}$$

where $p^\mu \equiv \dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A$. In fact, this shows that half of the components of each θ^A are left entirely unfixed by these equations. This is because the matrix $\Gamma \cdot p$ is nilpotent by the equations of motion, i.e. it squares to zero: $(\Gamma \cdot p)^2 = p^2 = 0$. This indicates that its rank is half of of its dimension $\mathcal{N}$. Since $\dot{\theta}^A$ appears in the action only in the combination $(\Gamma \cdot p)\dot{\theta}^A$, half of θ^A's components have no dynamics; they are are not physical propagating degrees of freedom, and therefore we have over estimated the fermionic content of our theory. The reason for this is something called κ symmetry, which the action (1.35) is invariant under. It may be expressed as

$$\delta\theta^A = i\Gamma \cdot p\,\kappa^A \qquad \delta x^\mu = i\bar{\theta}^A\Gamma^\mu\delta\theta^A \qquad \delta h = 4h\dot{\bar{\theta}}^A\,\kappa^A \tag{1.38}$$

where $\kappa^A(\tau)$ is a set of A *local* spinors; κ symmetry, unlike supersymmetry, is not global. This symmetry will be a necessary ingredient in the superstring action in order to ensure its supersymmetry.

The superstring action may be constructed for flat target spacetime in much the same way that the superparticle action was found. Generalizing to a non-flat target is an extremely non-trivial exercise which we won't discuss here. The flat space action may be expressed as

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{h}\, h^{ab}\, \Pi_a\cdot\Pi_b + S_\kappa \tag{1.39}$$

where $\Pi_a^\mu \equiv \partial_a X^\mu - i\bar{\theta}^A\Gamma^\mu\partial_a\theta^A$, and S_κ is a term which must be added in order to enforce the local κ symmetry. In fact, it turns out that this symmetry cannot be realized for arbitrary $\mathcal{N}$, we must take the number of supersymmetries to be ≤ 2. We will present S_κ for $\mathcal{N}=2$; the other cases may be obtained by setting one or both the θ^A's to zero. The form of S_κ involves coupling of the bosonic and fermionic degrees freedom, as well as a four-Fermi term

$$S_\kappa = T\int d^2\sigma\Big(-i\epsilon^{ab}\partial_a X^\mu\left(\bar{\theta}^1\Gamma_\mu\partial_b\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_b\theta^2\right) + \epsilon^{ab}\bar{\theta}^1\Gamma^\mu\partial_a\theta^1\,\bar{\theta}^2\Gamma_\mu\partial_b\theta^2\Big). \tag{1.40}$$

This addition must also obey the global $\mathcal{N}=2$ supersymmetry. This ends up setting constraints on the type of spinor θ may be, and on the spacetime dimension d. There are four choices involving $d=3,4,6$, and 10. We will see in the next section that only the $d=10$ choice will lead to a consistent quantum theory.

1.4.5 Light-cone gauge quantization and critical dimension

Our acquaintance with the superstring thus far has shown us some remarkable features. First the classical superstring can only have $\mathcal{N}\leq 2$. Second, in order for this to be true the dimension of the target spacetime must be 3, 4, 6, or 10. This power of the string to set parameters was one of the early attractions of the theory - it was hoped that the superstring would give a theory of everything where the number of parameters, or true "constants of nature", would be minimal. We will now see that the discretion of quantum mechanics goes further in this direction and requires the spacetime dimension to be 10.

The recipe for quantization is to replace Poisson brackets of fields and their conjugate momenta with commutators. In this way mode amplitudes (c-numbers) are promoted to operators which act upon a vacuum to create and annihilate states. If all of the gauge freedom is used-up prior to quantization, then one is guaranteed to have every state (made by acting the creation operators on the vacuum) be physical. The light-cone gauge quantization presented here is such a regime. We begin by fixing the gauge freedom afforded us by our superstring action (1.39). The κ symmetry allows us to set

$$\Gamma^+\theta^1 = \Gamma^+\theta^2 = 0 \tag{1.41}$$

where, as in section 1.4.2, $\Gamma^\pm = (\Gamma^0 \pm \Gamma^9)/\sqrt{2}$. As the matrices $\Gamma^\pm$ are nilpotent, this gauge choice fixes-out exactly half of the components of the spinors - as we saw in section 1.4.4, this is exactly the rôle of κ symmetry. Because, for $d=10$, the spinors θ^A are Majorana-Weyl, the 32 complex degrees of freedom associated to a generic spinor in $d=10$ are reduced to 16 real degrees of freedom. The additional constraint (1.41) reduces this further to 8 real components per spinor. Thus, in this gauge, the θ^A constitute an eight dimensional spinor representation of the group $SO(8)$. We showed in section 1.4.2 that a similar reduction occurs for the embedding coordinates X^μ. There the $X^\pm$ were fixed by gauge symmetry, leaving only the eight fields X^i as physical degrees of freedom. Taking these gauge choices for the θ^A and $X^\pm$, we find that the equations of motion resulting from (1.39) are immensely simplified

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^i = 0 \qquad (\partial_t + \partial_\sigma)\,\theta^1 = 0 \qquad (\partial_t - \partial_\sigma)\,\theta^2 = 0. \tag{1.42}$$

Notice that θ^1 is exclusively left-moving while θ^2 is exclusively right-moving. We have seen the mode expansions for the X^i in (1.18). For our fermionic partners, for closed superstrings, we have

$$\theta^{1\alpha}(\sigma,\tau) = \sum_{n=0}^{\infty} \beta_n^\alpha e^{-2in(\tau-\sigma)} \qquad \theta^{2\alpha}(\sigma,\tau) = \sum_{n=0}^{\infty} \widetilde{\beta}_n^\alpha e^{-2in(\tau+\sigma)} \tag{1.43}$$

where α, the $SO(8)$ spinor index, has been restored to emphasize that the fermionic oscillators β and $\widetilde{\beta}$ are spinors. The Poisson brackets between the fields and their conjugate momenta are

$$\begin{aligned} [\dot{X}^i(\sigma,\tau), X^j(\sigma',\tau)]_{\text{P.B.}} &= \pi\delta(\sigma-\sigma')\delta^{ij} \\ \{\theta^{A\alpha}(\sigma,\tau), \theta^{B\beta}(\sigma',\tau)\}_{\text{P.B.}} &= i\pi\delta^{\alpha\beta}\delta^{AB}\delta(\sigma-\sigma'), \end{aligned} \tag{1.44}$$

this implies

$$\begin{aligned} [\alpha_m^i, \alpha_n^j]_{\text{P.B.}} = im\delta_{m+n}\delta^{ij} \qquad [\widetilde{\alpha}_m^i, \widetilde{\alpha}_n^j]_{\text{P.B.}} = im\delta_{m+n}\delta^{ij} \qquad [\alpha_m^i, \widetilde{\alpha}_n^j]_{\text{P.B.}} = 0 \\ \{\beta_m^\alpha, \beta_n^\beta\}_{\text{P.B.}} = im\delta_{m+n}\delta^{\alpha\beta} \qquad \{\widetilde{\beta}_m^\alpha, \widetilde{\beta}_n^\beta\}_{\text{P.B.}} = im\delta_{m+n}\delta^{\alpha\beta} \qquad \{\beta_m^\alpha, \widetilde{\beta}_n^\beta\}_{\text{P.B.}} = 0 \end{aligned} \tag{1.45}$$

Quantization amounts to the replacement $[A,B]_{\text{P.B.}} \to i[A,B]$, and $\{A,B\}_{\text{P.B.}} \to i\{A,B\}$, which implies that the oscillators are promoted to creation and annihilation operators.

We have alluded to the fact that quantization selects for us a target spacetime dimension d. In fact this selection can be seen in an anomaly arising in the Lorentz group algebra (1.31). The gauge choice (1.25) explicitly breaks Lorentz invariance by choosing a preferred direction. Under quantization, those elements of $M_{\mu\nu}$ which mix the $+$ direction with the others can and do develop an anomaly. Specifically, the commutator $[M^{i-}, M^{j-}]$, which must be zero classically, is no longer so after quantization. The proof of this proceeds as follows: first $M^{\mu\nu}$ is constructed using the standard method

$$M^{\mu\nu} = T\int_0^\pi d\sigma \left(X^\mu\dot{X}^\nu - X^\nu\dot{X}^\mu + \bar{\theta}^A\Gamma^{\mu\nu}\theta^A\right). \tag{1.46}$$

The mode expansions are then inserted, giving an expression in terms of creation and annihilation operators. The algebra (1.31) is then evaluated using the (quantum versions of the) commutation relations (1.45). The end result is that $[M^{i-}, M^{j-}] = 0$ if and only if the target spacetime dimension is $d = 10$, while the rest of the algebra (including the supersymmetric extension) is anomaly free.

1.4.6 Closed string spectrum, background fields, and low energy effective actions

The superstring has selected for us the amount of supersymmetry and the spacetime dimension d. The next natural question to ask is what the particle content of the theory is, and what the interactions are between those particles. Again the notion of a quantum anomaly is important here. We saw in section 1.4.1 that the string worldsheet possessed conformal or Weyl invariance. Should we place our superstring in a general target space background (a given metric, and possibly other gauge fields and superpartners), we will generically develop a quantum conformal anomaly on the worldsheet. The requirement that this anomaly vanish gives us equations of motion for the background fields which are exactly those obeyed by the string modes themselves - i.e. a string will develop a conformal anomaly unless it is placed in a background of strings. Thus, not only does the superstring choose its supersymmetry and spacetime dimension, but also tells us that everything is made of string. This is a further manifestation of self-reference and internal consistency. It therefore seemed to early researchers that superstring theory could indeed be a theory of everything.

The closed string spectrum is generated by tensoring the left and right moving modes, which are representations of the $SO(8)$ symmetry enjoyed by (1.39). The bosonic modes of the X^i are obviously in the vector representation $\mathbf{8_v}$, whereas the fermionic modes of the θ^A are $SO(8)$ spinors which come in two chiralities on account of them being Weyl; these are labelled as $\mathbf{8_c}$ and $\mathbf{8_s}$. The lowest energy (massless) states of the string theory correspond to two-mode excitations (one right and one left moving, since for closed strings the number of left and right-movers must be equal, see (1.29)). We are free to take the left-moving and right-moving modes to have same or opposite spinor chirality; the choice will lead to different string theories. If we take them to be opposite, the following massless spectrum is generated

$$\begin{aligned}(\mathbf{8_v}+\mathbf{8_c})_L\otimes(\mathbf{8_v}+\mathbf{8_s})_R = &(\mathbf{1}+\mathbf{28}+\mathbf{35_v}+\mathbf{8_v}+\mathbf{56_v})_B\\ &+(\mathbf{8_s}+\mathbf{8_c}+\mathbf{56_s}+\mathbf{56_c})_F\end{aligned} \tag{1.47}$$

where the subscript B stands for bosons and F for fermions. This is the spectrum of type IIA supergravity. If we take the same chirality for left and right-movers, we obtain

$$\begin{aligned}(\mathbf{8_v}+\mathbf{8_c})_L\otimes(\mathbf{8_v}+\mathbf{8_c})_R = &(\mathbf{1}+\mathbf{28}+\mathbf{35_v}+\mathbf{1}+\mathbf{28}+\mathbf{35_c})_B\\ &+(\mathbf{8_s}+\mathbf{8_s}+\mathbf{56_s}+\mathbf{56_s})_F\end{aligned} \tag{1.48}$$

which is the spectrum of type IIB supergravity. The lesson is that if we restrict ourselves to the least excited strings, those with the smallest energy (which happen to be massless), we obtain the particle content of something called supergravity, a theory we will explain below. Indeed, the interactions between these string modes are also identical to the supergravity interactions, leading us to the conclusion that the low-energy effective dynamics of closed superstring theory is supergravity.

Supergravity is a supersymmetrization of Einstein gravity. Unlike the global supersymmetry of field theories, in supergravity the supersymmetry is promoted to a local symmetry whose gauge connection is an object of spin 3/2. It is believed that massless particles of spin > 2 cannot be coupled consistently in any field theory. This fact places an upper limit on the spacetime dimension a supergravity theory may live in. It turns out that if $d > 11$, local supersymmetry requires the presence of massless particles whose spin is greater than two. Therefore, 11-dimensional supergravity plays a privileged rôle, and some lower dimensional supergravities may be realized through toroidal compactification of this theory. We will present only the bosonic content of the supergravities, as the fermionic content can be obtained from supersymmetry. The 11-dimensional supergravity contains a spacetime metric G_{MN} and a 3-form gauge potential A_3. Its action is as follows

$$S_{11}^{\text{bos.}} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4 \tag{1.49}$$

where R is the Ricci scalar built from G_{MN} and $F_4 \equiv dA_3$. Compactifying one direction of this theory with period $2\pi R$ causes the 11-dimensional metric to be mapped to a 10-dimensional scalar Φ, vector A_1, and traceless symmetric tensor (metric) G_{mn}. The 3-form is mapped to another 3-form (we'll keep the label A_3) and a 2-form B_2. A glance at (1.47) reveals precisely this pattern - the $\mathbf{8_v}$ are the physical propagating degrees of freedom corresponding to a massless vector in ten dimensions, the $\mathbf{28}$ is that corresponding to an antisymmetric rank-2 tensor (i.e. B_2), etc. This is type IIA supergravity, whose action may be written as

$$\begin{aligned} S_{\text{IIA}}^{\text{bos.}} = & \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ & -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(|F_2|^2 + |\widetilde{F}_4|^2\right) - \frac{1}{4\kappa_{10}^2}\int B_2 \wedge F_4 \wedge F_4 \end{aligned} \tag{1.50}$$

where $F_{M+1} \equiv dA_M$, $H_3 = dB_2$, $\widetilde{F}_4 = dA_3 - A_1 \wedge H_3$, $\kappa_{10}^2 = \kappa_{11}^2/2\pi R$, and we have rescaled the metric by the factor $\exp(-2\Phi/3)$. The field Φ, called the dilaton, plays a very important rôle here. This is because the effective coupling of the theory, i.e. the 10-dimensional universal constant of gravitation is given by $8\pi G_{10} = (\kappa_{10}e^{\Phi})^2$. Therefore the dilaton sets the coupling strength - the coupling constant in string theory is *dynamical*[6]. This means that it does not need to be set as a parameter, the theory determines it for us self-consistently. The corresponding action for type IIB supergravity contains, instead of 1 and 3-form fields, 0, 2, and 4-form fields labelled C_0, C_2, and C_4. Its action may be written as

$$\begin{aligned} S_{\text{IIB}}^{\text{bos.}} = & \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ & -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2\right) - \frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3 \end{aligned} \tag{1.51}$$

where $F_{M+1} \equiv dC_M$, $\widetilde{F}_3 = F_3 - C_0 \wedge H_3$, $H_3 = dB_2$, and $\widetilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$.

[6]In general the closed string coupling is denoted by $g_s \equiv \exp(\langle\Phi\rangle)$. This is the quantity which weights the vertices where strings interact, analogous to α in Quantum Electrodynamics. The value of of κ_{10} may be determined via a closed string exchange, it is equal to $[(2\pi)^7\alpha'^4/2]^{1/2}$.

We have now seen the effective dynamics of closed superstrings when the energy scale is low enough not to excite massive string modes. The equations of motion following from the actions (1.50) and (1.51), and therefore the interactions between the particles of their fields, are recovered by superstring interactions. Further, we remind the reader that only when the superstring is placed in a background obeying these equations of motion will the worldsheet theory be free of the conformal anomaly. To a very large extent "string theory" is concerned with type IIA and type IIB supergravity; genuine perturbative stringy effects are difficult to calculate and therefore do not appear in any considerable volume in the literature. However, we shall see in the next section that open strings do give rise to tractable and immensely powerful non-perturbative objects which have no analogue in point-particle theories, the D-branes. It should be noted that there are two very important closed string theories, and one closed + open string theory that have not been presented here - the heterotic string theories and the type I superstring respectively. These theories may be obtained via various dualities which act upon the type II theories. Details of these string theories may be found in the standard textbooks [33–36].

1.4.7 Open strings, T-duality, and D-branes

It was realized in the late 1980's [3] that strings see the geometry of their target spacetime in a radically different way than point particles. It is easiest to see this from the point of view of closed strings. Consider the zero modes of the closed string (1.18) (i.e. set $\alpha_n^\mu = \widetilde{\alpha}_n^\mu = 0$) propagating on a flat target space which contains an S^1. Let the the radius of the S^1 be R and consider the zero mode of the embedding function in this direction. We have

$$X(\sigma,\tau) = \frac{1}{2}(x_L + x_R) + \alpha'(p_L + p_R)\,\tau + \alpha'(p_L - p_R)\,\sigma. \tag{1.52}$$

Imposing the closed string boundary conditions is different in this space because $X \sim X + 2\pi R$. Therefore we must only have that $X(\sigma,\tau) = X(\sigma+\pi,\tau) + 2\pi R w$, where $w \in \mathbb{Z}$ is called the winding (or wrapping) number as it counts the number of times the string winds the S^1 before closing back on itself. We can also note that quantum mechanical momentum on a circle is quantized in units of the inverse radius. Given these facts we see that (see (1.18))

$$p_L + p_R = \frac{2}{R}n, \qquad p_L - p_R = \frac{2R}{\alpha'}w \tag{1.53}$$

where $n \in \mathbb{Z}$ is the momentum quantum number[7]. Now consider the following target space transformation, $R \to \alpha'/R$. Under this operation the closed string zero mode simply sees an exchange of n and w, and nothing else. More precisely a closed string zero mode cannot tell whether it is propagating on a circle of radius R or α'/R. This effect, dubbed *Target space duality* or *T-duality* was shown to extend beyond the level of the zero modes [3] and is a symmetry of the full string theory. The T-dual operation, extended to all moments of the momenta, amounts to

[7]Note that the level matching condition (1.29) is now modified since $p_L \neq p_R$. The result is that nw is added to the RHS. Also note that the tension of the wound string and its momentum in the compact direction will now contribute to its mass (1.28).

$$\alpha_n \to -\alpha_n, \qquad \widetilde{\alpha}_n \to \widetilde{\alpha}_n \tag{1.54}$$

and so may be realized via the replacement $X(\sigma,\tau) \to X_L(\sigma,\tau) - X_R(\sigma,\tau)$, which is referred to as the *T-dual coordinate*.

It is instructive to consider what effect this operation has on open strings. The open string cannot wrap a compact direction because it does not close. Thus the periodicity of the target space does not effect the mode expansion, which is

$$X(\sigma,\tau) = x + 2\alpha' p\tau + i\sqrt{\frac{2}{\alpha'}}\sum_{n\neq 0}\frac{1}{n}\alpha_n\, e^{-in\tau}\cos(n\sigma) \tag{1.55}$$

where $\sigma \in [0,\pi]$. We are free to write this in terms of a sum of a function of $(\tau+\sigma)$ and one of $(\tau-\sigma)$, i.e. as a sum of left and right-moving pieces

$$\begin{aligned} X(\sigma,\tau) &= X_L + X_R \qquad \text{where} \\ X_L &= \frac{1}{2}x_L + \alpha'\frac{n}{R}(\tau+\sigma) + \frac{i}{\sqrt{2\alpha'}}\sum_{n\neq 0}\frac{1}{n}\alpha_n\, e^{-in(\tau+\sigma)} \\ X_R &= \frac{1}{2}x_R + \alpha'\frac{n}{R}(\tau-\sigma) + \frac{i}{\sqrt{2\alpha'}}\sum_{n\neq 0}\frac{1}{n}\alpha_n\, e^{-in(\tau-\sigma)}. \end{aligned} \tag{1.56}$$

What does this open string embedding look like in the T-dual theory? The T-dual coordinate is

$$X_L - X_R = x_0 + 2\alpha'\frac{n}{R}\sigma + \sqrt{\frac{2}{\alpha'}}\sum_{n\neq 0}\frac{1}{n}\alpha_n\, e^{-in\tau}\sin(n\sigma). \tag{1.57}$$

There are two things to realize about this embedding. First the endpoints of the open string ($\sigma = 0,\pi$) do not oscillate (the sin function is zero there) and second, they are fixed at x_0 and $x_0 + 2\pi n\widetilde{R}$, where $\widetilde{R}$ is the T-dual radius α'/R. In the T-dual theory these points are identified in the target space. The embedding functions in the other directions are unaffected by this of course, so we have an open string whose bulk fluctuates in the full spacetime but whose endpoints are confined to a $d-1$ dimensional hyperplane. If we take in addition $R \to 0$, the bulk of the string sees a (T-dualized) compact direction of infinite radius. What we have found is a *Dirichlet (d-2)-brane* or *D(d-2)-brane*. This is displayed in figure 1.9. We did not discuss open superstrings in section 1.4.6, however there is an open supersymmetric string theory called type I, which includes both open and closed strings and is unoriented. If we T-dualize any odd number p of dimensions we will end up with a D$(10-p-1)$-brane and type IIA closed strings far away from the brane. If we T-dualize any even number of dimensions q, we will end up with a D$(10-q-1)$-brane and type IIB closed strings far away from the brane[8]. What is the theory governing the strings on these branes? The bosonic

[8]This is not precisely true, as things are complicated by the fact that type I theory has an $SO(32)$ gauge group. In fact, multiple D-branes are produced upon T-dualizing type I, as well as objects called *orientifold planes*, a consequence of the unoriented nature of type I strings. See [34], pg. 138 for details.

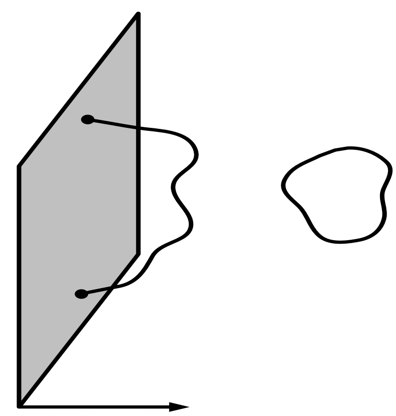


Figure 1.9: A D-brane supports the endpoints of open strings, which are free to move in the brane's worldvolume. The bulk of the open string is not confined, nor are closed strings propagating in the spacetime.

degrees of freedom of the Dp brane are described by a $(p+1)$-dimensional gauge field A_a, corresponding to string excitations in the worldvolume of the brane and $10-(p+1)$ scalars Φ_I describing transverse string excitations. The vacuum expectation values (VEV's) or zero modes of these fields describe the embedding of the brane into the target space, $X^\mu(\xi^a)$, where the ξ_a are the worldvolume coordinates of the brane. Therefore the X^I describe the shape of the brane while the X^a describe any constant gauge field backgrounds turned-on on the worldvolume. The bosonic action is given by the Dirac-Born-Infeld [24] or DBI action[9]

$$S_{Dp} = -T_p \int d^{p+1}\xi\, e^{-\Phi} \left[-\det\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right)\right]^{1/2} + i\mu_p \int_{p+1} \exp\left(2\pi\alpha' F_2 + B_2\right) \wedge \sum_q C_q \tag{1.58}$$

where

$$G_{ab} = G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \qquad B_{ab} = B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu = P[B_2] \tag{1.59}$$

i.e. the metric and antisymmetric B-field from the target space are pulled back onto the worldvolume of the D-brane. Here F_{ab} is the field strength built on A_a, i.e. $F_{ab} = \partial_a A_b - \partial_b A_a$. Note that Φ above is the dilaton (from (1.50) for example) and not the transverse scalars. Note also the appearance of the space-time form potentials C_q; the D-branes carry charge μ_p under these potentials. Note that the expansion of the exponential will give forms of various rank, but the integral will only be non-zero for those combinations which amount to a $(p+1)$-form. The D-brane charge and tension are calculable via a closed string exchange amplitude. The result is that $T_p = \mu_p = (2\pi)^{-p}(\alpha')^{-(p+1)/2}$. However, since we have a

[9] The second term involving the coupling to the form potentials C_M is called the *Chern-Simons* [25] term.

factor of $e^{-\Phi}$ in front of (1.58), the effective D-brane tension is inversely proportional to the string coupling (see discussion beneath (1.50)). Therefore the D-brane is a non-perturbative object, infinitely heavy at zero coupling. We could never have hoped to discover it through perturbative techniques.

1.5 AdS/CFT correspondence

> **Glaucon**: You have shown me a strange image, and they are strange prisoners.
> **Socrates**: Like ourselves, I replied; and they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave?[10]
> — Plato's *The Republic*, Book VII

The AdS/CFT correspondence, in its most celebrated form, is a conjectured duality between type IIB string theory on the background space $AdS_5 \times S^5$ (with a background 4-form potential), and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four spacetime dimensions. There are many other manifestations of this duality, which is really a much deeper statement about the connection between gauge theories and gravity. It is also an instance of *holography*, where a higher dimensional gravity is entirely captured by a lower dimensional quantum field theory. That gravity has this suspicious scaling of its physical degrees of freedom was hinted at in the 1970's by Beckenstein [5], who associated the area of a black hole's horizon with an entropy. This picture was later strengthened by Hawking's discovery [6] that semi-classically, black holes produce a thermal spectrum of radiation, whose characteristic temperature T is related to the surface gravity κ of the black hole via $T = \kappa/2\pi$. Indeed the four laws of thermodynamics may be applied to the black hole with these identifications [7]. The situation was much improved upon the successful microscopic computation of the black hole entropy using string theory [8]. This calculation depended upon the concept that fluctuation modes of D-branes which served as the central "mass" of the black holes, embodied the microstates responsible for the macroscopic, spacetime entropy. The emerging duality between the brane dynamics and those of the curved spacetime which they source, including the realization that absorption cross sections could be calculated from either perspective [9–11], led Maldacena [13] to the AdS/CFT correspondence in 1997. The significance of this discovery is twofold. On the one hand it offers insights into gravity via quantum field theory. On the other it affords the long sought-after string description of (at least some) gauge theories, and their strong coupling dynamics. In fact, Polyakov [4] had already realized that the string description of gauge theories required the string to propagate in a higher dimensional spacetime, before the Maldacena conjecture appeared. We will give below a general introduction to the AdS/CFT correspondence, its main features, and a cross section of results pertinent to this thesis.

1.5.1 Supergravity p-branes and string theory D-branes

Before the discovery of D-branes, solutions of supergravity were discovered which were solotonic hyperplanes [14–16]. These solutions, called *p-branes*, exist in both type IIA and type IIB supergravities, their general form given by

[10] It was Polyakov [4] who originally noted the appropriateness of the classic allegory to holography.

$$
\begin{aligned}
ds^2 = H^{-1/2}(r)\left[-f(r)dt^2 + (dx^i)^2\right] + H^{1/2}(r)\left[f^{-1}(r)dr^2 + r^2 d\Omega_{8-p}^2\right],\\
e^{\Phi} = H^{\frac{(3-p)}{4}}(r), \qquad F_{ti_1\ldots i_p r} = \epsilon_{i_1\ldots i_p}\frac{1}{H^2(r)}\frac{Q}{r^{8-p}},\\
H(r) = 1 + \left(\frac{R}{r}\right)^{7-p}, \qquad f(r) = 1 - \left(\frac{r_0}{r}\right)^{7-p}
\end{aligned}
\tag{1.60}
$$

where $F_{ti_1\ldots i_p r}$ is the $(p+2)$-form field strength from (1.50) or (1.51), so that p must be odd for type IIB solutions and even for type IIA. Q is the charge of the solution under this form field. In general these solutions have horizons at $r = r_0$, and hence are extended black hole solutions. Unlike the standard Schwarzschild black hole [17], whose singularity is point-like, here the singularity is extended in a p-spatial-dimensional hyperplane, covered by the coordinates x_i. These solutions are also charged, and so may be viewed as generalizations of the Reissner-Nordström solution [18]. Like that solution, there is a bound relating the mass and charge $M \geq Q$, both of which are functions of R and r_0; when this bound is saturated $r_0 = 0$, and the solution is called *extremal*. In equation (1.58), and the discussion beneath it, we saw that D-branes carry a charge μ_p which is equal to their tension or "mass" T_p. It should not be surprising that the flat Dp-branes are extremal p-brane solutions, viewed in the low energy limit where the supergravity description is appropriate. In fact the equality of mass and charge is a reflection of supersymmetry; the D-brane (or extremal p-brane) preserves 1/2 of the 32 supersymmetries of the original closed string theory (or supergravity). This is often referred to as 1/2 BPS, where BPS is named after Bogomol'nyi, Prasad, and Sommerfield [19, 20]. The case of the extremal 3-brane is the most important for the AdS/CFT correspondence. In fact, in the realm of 10-dimensional theories, only the 3-brane will give Anti-deSitter or AdS space in a given limit; conversely only AdS space will have a conformal theory on its boundary[11].

Setting $p = 3$ and $r_0 = 0$, we arrive at the following solution for the extremal 3-brane

$$
ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2}\left(-dt^2 + dx_i^2\right) + \left(1 + \frac{R^4}{r^4}\right)^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right). \tag{1.61}
$$

We note that the dilaton is constant, and so the string coupling is the same everywhere. There is also a self-dual five-form given by

$$
F_5 = (1 + *)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d\left(\left[1 + \frac{R^4}{r^4}\right]^{-1}\right). \tag{1.62}
$$

The first order of business is to relate the parameters of the D3-brane to this solution. In fact, we will be interested in a stack of N parallel D3-branes. We are free to do this because parallel D-branes do not interact with each other - a consequence of supersymmetry which ensures that their gravitational attraction is balanced exactly by their "electro-magnetic" (in the sense of the form potentials) repulsion. The first thing to do is to equate the tension

[11] We will discuss 11-dimensional versions of AdS/CFT in chapter 3. It should also be mentioned that there is a version of AdS/CFT dealing with the space $AdS_3 \times S^3 \times M^4$ where M^4 is a compact manifold. This theory will not be discussed in this thesis.

of the N D3-branes to the ADM [21] mass of the spacetime (1.61). The ADM mass is the general relativistic measure of the stress energy responsible for the curvature of spacetime; it is the gravitational "charge". The ADM mass has been calculated in [22], the result is

$$M_{ADM} = \frac{2\pi^3}{8\pi G_{10}} R^4 = \frac{R^4}{32\pi^4 g_s^2 \alpha'^4} \tag{1.63}$$

where we have used $8\pi G_{10} = (\kappa_{10}\, g_s)^2$ and $\kappa_{10}^2 = (2\pi)^7 \alpha'^4/2$ in the second equality, as per section 1.4.6. The D-brane tension was given in section 1.4.7, multiplying this by N we have

$$\tau_{ND3} = g_s^{-1} T_{ND3} = \frac{N}{8\pi^3 g_s \alpha'^2}. \tag{1.64}$$

where we have noted the dilaton factor in (1.58). Equating (1.63) and (1.64), we arrive at a special relation

$$\boxed{R^4 = 4\pi g_s N \alpha'^2}\,. \tag{1.65}$$

So far we have analyzed the N D3-branes in terms of the low energy supergravity description. Recall that this is the picture seen by closed strings propagating in the bulk, see figure 1.9. We should also ask ourselves what the open strings attached to the D-brane are doing. To answer this we take the low energy limit ($\alpha' \to 0$) of the DBI action (1.58) describing our D3-brane. We have no B_2 field in the background, and the dilaton Φ is a constant defining g_s. Further, the D-brane is in flat 10-dimensional space, and so $G_{\mu\nu} = \eta_{\mu\nu}$. We take the embedding to be as follows

$$X^a(\xi^a) = \xi^a, \qquad X^I(\xi^a) = \sqrt{2}\pi\alpha' \Phi^I(\xi^a) \tag{1.66}$$

where $a = 0, \ldots, 3$ are the worldvolume coordinates of the D3-brane and $I = 4, \ldots, 9$ are the transverse directions. Ignoring the coupling to the space-time form potentials, we have

$$S_{D3} = -\frac{1}{(2\pi)^3 g_s \alpha'^2} \int d^4\xi \sqrt{-\det\left(\eta_{ab} + 2\pi\alpha' F_{ab} + (\sqrt{2}\pi\alpha')^2 \partial_a \Phi^I \partial_b \Phi^I\right)} \tag{1.67}$$

where we have explicitly indicated the D3-brane tension. Expanding to leading order in α', we have

$$S_{D3} = -\frac{V_4}{(2\pi)^3 g_s \alpha'^2} - \frac{1}{4\pi g_s} \int d^4\xi \left(\frac{1}{4}(F_{ab})^2 + \frac{1}{2}(\partial_a \Phi^I)^2\right) + \ldots \tag{1.68}$$

where V_4 is the infinite volume of the brane. Apart from this constant, we have a free gauge theory with six scalars. Our task is not quite this however, since we would like to describe the worldvolume theory of open strings on N coincident, parallel D3-branes. The generalization required is not difficult to understand, see figure 1.10. Open strings are free to begin and end on any of the N branes, without penalty in energy since the branes are coincident. In the $N = 1$ case, the modes of a string parallel to the brane described a gauge field $A^a(\xi^a)$. Now this field has a factor of N^2 times the number of components in order to allow for the specification of the string end-points' branes-of-residence. Thus $A^a(\xi^a)$ is promoted to an

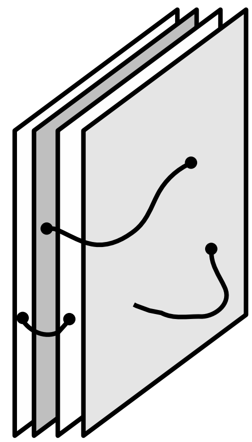


Figure 1.10: A stack of N D-branes, intended to be coincident as well as parallel, but shown separated for clarity. Open strings may begin and end on any two (or the same) branes, without penalty in energy. The effect is that worldvolume fields $X^\mu(\xi^a)$ are promoted to $N \times N$ matrices, leading to a non-abelian supersymmetric Yang-Mills theory.

$N \times N$ unitary matrix, and similarly for the scalars describing the transverse position of the stack. The full generalization of the action (1.58) to this case is known [23], but rather than indicating it explicitly here, we give the $\alpha' \to 0$ limit of it for the D3-brane, i.e. the generalization of (1.68)

$$S_{ND3} \simeq -\frac{1}{4\pi g_s} \mathrm{Tr} \int d^4x \left(\frac{1}{2}(F_{\mu\nu})^2 + (D_\mu \Phi^I)^2 - \frac{1}{2}[\Phi^I, \Phi^J]^2 + \text{fermions} \right) + \dots \qquad (1.69)$$

where the trace is over the $U(N)$ matrix indices, we have changed the worldvolume coordinates to x_μ, and the leading constant proportional to volume has been dropped. Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and $D_\mu \Phi^I = \partial_\mu \Phi^I - i[A_\mu, \Phi^I]$. We have indicated "+ fermions" to remind the reader that even equation (1.58) is only the bosonic portion of the action. All of these objects are supersymmetric and so fermions must be added in the appropriate manner. The action (1.69) is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four space-time dimensions. It is a non-abelian $U(N)$ gauge theory, as we encountered in section 1.3, see equation (1.5). Comparing the forms of the action, we arrive at the second fundamental relation of the AdS/CFT correspondence

$$\boxed{4\pi g_s = g_{YM}^2} \qquad (1.70)$$

i.e. the Yang-Mills coupling constant is related to the square-root of the closed string coupling.

1.5.2 Absorption cross-sections

We have seen that a stack of N D3-branes, seen from the low-energy supergravity limit, i.e. away from the branes (see figure 1.9), looks like a curved spacetime (1.61) with a five-form field strength turned on. We have also seen that the low-energy limit of the theory on the stack of D-branes is $\mathcal{N} = 4$ supersymmetric Yang-Mills with gauge group $U(N)$. One of the indications that these two pictures might be equivalent came from the consideration of the absorption of closed string modes by either the geometry (1.61) or by the stack of D-branes [9–11]. The geometry may be envisioned as having a central throat from which it is difficult for particles to escape, see figure 1.11. We begin by considering the absorption of

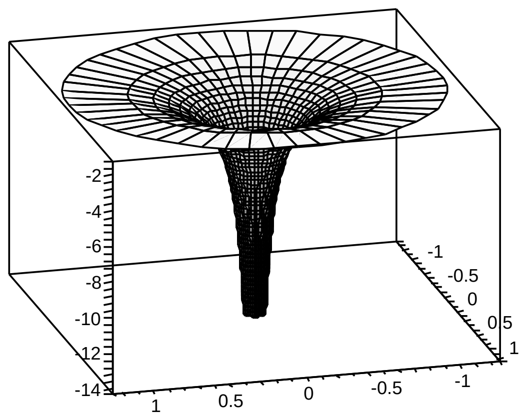


Figure 1.11: The ratio of proper radial distance to coordinate radial distance $dr_P/dr = (1 + R^4/r^4)^{1/4}$ is plotted on the vertical axis for the extremal 3-brane geometry (1.61). The angular variable may be thought of as representing the five-sphere. The worldvolume coordinates x_i are suppressed. Here R is set to one and dr_P/dr is multiplied by -1 for visual effect.

the fluctuations of the dilaton Φ, which we will call ϕ. The equation of motion for this field can be obtained from (1.51), using the solution (1.61), (1.62). It turns out that it is simply $\Box\phi = 0$, where the D'Alembertian $\Box$ is defined by the metric (1.61). We take the following form for ϕ

$$\phi(X) = R(r)\Theta(\Omega_5)e^{-i\omega t} \tag{1.71}$$

where X indicates the full 10-dimensional coordinates, r is the radial coordinate from (1.61) and Ω_5 is shorthand for the coordinates on the five-sphere. The energy of the fluctuation is given[12] by ω. Notice that we have suppressed dependence on the brane coordinates x_i; we will not be interested in these fluctuations as they will not contribute to the absorption cross-section. It is straightforward to calculate the D'Alembertian, which then gives the following equation of motion

[12]As is customary, the units chosen in this thesis are such that $\hbar = c = 1$.

$$\Box\phi = \left[-\left(1+\frac{R^4}{r^4}\right)\partial_t^2 + \partial_r^2 + \frac{5}{r}\partial_r + r^2 D^2_{\Omega_5}\right]\phi(X) = 0 \tag{1.72}$$

where $D^2_{\Omega_5}$ indicates the Laplacian on the five-sphere. We will consider only the s-wave or $D^2_{\Omega_5}\Theta = 0$ modes, and hence calculate the s-wave absorption cross-section. The resulting radial equation is

$$\left[\omega^2\left(1+\frac{R^4}{r^4}\right) + r^{-5}\partial_r(r^5\partial_r)\right]R(r) = 0. \tag{1.73}$$

It is simpler to solve this equation after the following change of variables $r = Re^{-z}$, $R(r) = e^{2z}\psi$, then

$$\left[\partial_z^2 + 2\omega^2 R^2 \cosh 2z\right]\psi(z) = 0 \tag{1.74}$$

and our problem reduces to a Schrödinger equation with potential $-2\omega^2R^2\cosh 2z$. This is a barrier problem where the incoming wave (from $z = -\infty$) has zero "energy" and the top of the potential is also at zero energy. Thus ψ is on the border between tunnelling or conventionally transmitting from $z = -\infty$ to $z = \infty$, i.e. from asymptotic flat space at $r = \infty$ to the center of the throat. The equation may be solved easily in the $z \to \infty$ and in the $z \to -\infty$ limits, the solutions are as follows

$$\psi_{-\infty}(z) \simeq aJ_2(\omega Re^{-z}), \qquad \psi_\infty(z) \simeq iH_2^{(1)}(\omega Re^{z}) \tag{1.75}$$

where $H_m^{(1)}$ is the m^{th} Hankel function of the first kind, and J_2 is the second Bessel function. In the region $\ln\omega R \ll z \ll -\ln\omega R$, i.e. for low energies ω, the two solutions are simultaneously valid, see figure 1.12. This allows for the determination of the in-going "amplitude" a, by matching the two solutions in their overlapping region. Investigating the asymptotics

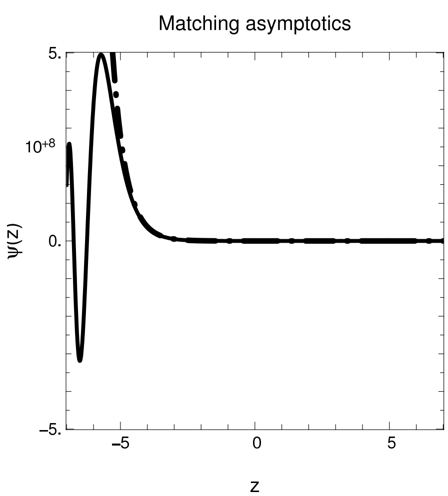


Figure 1.12: The solutions (1.75) are plotted for $\omega R = 0.01$, where the solid line is $\psi_{-\infty}(z)$ and the dot-dash the real part of $\psi_\infty(z)$ (the imaginary part is negligible). It is seen that they overlap in the region $\ln\omega R \ll z \ll -\ln\omega R$.

for $z = 0$ and $\omega R \ll 1$, we find

$$J_2(\omega R e^{-z}) \simeq \frac{(\omega R)^2}{8}, \qquad iH_2^{(1)}(\omega R e^{z}) \simeq \frac{4}{\pi(\omega R)^2} \tag{1.76}$$

and so

$$a = \frac{32}{\pi}(\omega R)^{-4}. \tag{1.77}$$

The opposite asymptotics give

$$\begin{aligned} aJ_2(\omega R e^{-z}) &\simeq -\sqrt{\frac{2}{\pi\omega R e^{-z}}}\frac{a}{2}\left(e^{i(\omega R e^{-z}-\pi/4)} + e^{-i(\omega R e^{-z}-\pi/4)}\right), \qquad z \to -\infty \\ iH_2^{(1)}(\omega R e^{z}) &\simeq -i\sqrt{\frac{2}{\pi\omega R e^{z}}}e^{i(\omega R e^{z}-\pi/4)}, \qquad z \to \infty \end{aligned} \tag{1.78}$$

from which we can read-off the incident, reflection, and transmission coefficients. The absorption probability is the squared norm of the ratio of the transmission coefficient to the incident coefficient, or

$$\mathcal{P} = \left|\frac{1}{a/2}\right|^2 = \frac{\pi^2}{16}(\omega R)^8 \tag{1.79}$$

The task of translating this into a cross-section is rather involved in the general case [26], the result is the following prescription

$$\sigma = \frac{(2\pi)^n}{\omega^n \Omega_n}\mathcal{P} = \frac{\pi^4}{8}\omega^3 R^8 \tag{1.80}$$

where n is the dimension of the sphere in the geometry, in our case $n = 5$, while $\Omega_n = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$ is the volume of the n-sphere. We have therefore found the absorption cross-section for the dilaton s-waves in the extremal 3-brane geometry.

How do we envision this process from the point of view of the stack of D3-branes? Although we made no mention of it, the action (1.58) clearly contains a coupling to the dilaton, i.e. the e^{Φ} factor. We may therefore ask the question, what is the total cross-section for dilaton absorption by a stack of N D3-branes? In order to answer this question we should begin by analyzing the low-energy limit of our D3-branes and the subsequent dilaton coupling. This is most easily accomplished by placing the action (1.51) into canonical form by rescaling the metric

$$G_{\mu\nu} \to \widetilde{G}_{\mu\nu} = e^{\frac{1}{2}(\Phi_0 - \Phi)}G_{\mu\nu} \tag{1.81}$$

where the dilaton has been shifted by a constant Φ_0. We will be interested only in the action for the canonically normalized dilaton $\widetilde{\Phi} = \Phi - \Phi_0$. Applying the rescaling to (1.51), we have[13]

[13]Note that $\widetilde{R} = e^{\widetilde{\Phi}/2}\left[R - \frac{9}{2}\nabla^2\widetilde{\Phi} + \frac{9}{2}\partial_\mu\widetilde{\Phi}\partial^\mu\widetilde{\Phi}\right]$, see for example Appendix E of [27].

$$S_{10} = \frac{1}{2(\kappa_{10}e^{\Phi_0})^2}\int d^{10}x\sqrt{-\widetilde{G}}\left(\widetilde{R} - \frac{1}{2}\partial_\mu\widetilde{\Phi}\partial^\mu\widetilde{\Phi} + \ldots\right). \tag{1.82}$$

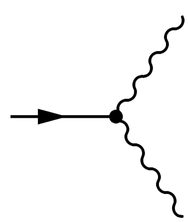
This frame is referred to as the *Einstein frame*, while the pre-scaled version is dubbed the *string frame*. Notice that the gravitational coupling is determined by the constant dilaton shift, $\widetilde{\kappa}_{10} = \kappa_{10}e^{\Phi_0} = \kappa_{10}g_s$. Shifting to the Einstein frame in our D-brane action (1.58), we see that only the G_{ab} term is affected. The result for the low-energy action of our stack of D3-branes is

$$S_4 = -\frac{1}{2g_{YM}^2}\,\mathrm{Tr}\int d^4x\left(e^{-\widetilde{\Phi}}(F_{\alpha\beta})^2 + \ldots\right) \tag{1.83}$$

where the "..." refers to terms not coupled to the dilaton and fermion terms[14]. Also, we have omitted couplings to the dilaton involving the transverse scalars on the D-brane; these correspond to higher-than-s-wave dilaton couplings. We now take $\widetilde{G}_{\mu\nu} = \eta_{\mu\nu}$ since we take the D3-branes to be sitting in flat ten-dimensional space. Further, we put the dilaton action (1.82) into standard canonical form by defining $\phi = \widetilde{\Phi}/\sqrt{2}\widetilde{\kappa}_{10}$; we do the same for the coupling in (1.83) by rescaling A_α by g_{YM}. We then obtain

$$S_{10} + S_4 = -\frac{1}{2}\int d^{10}x\,\partial_\mu\phi\,\partial^\mu\phi - \int d^4\xi\left(\frac{1}{4}(F^a_{\alpha\beta})^2 + \frac{\widetilde{\kappa}_{10}}{\sqrt{2}}\,\phi\,\partial_\alpha A^a_\beta\,\partial^\alpha A^{a\beta}\right) \tag{1.84}$$

where we have used the fact that the generators T^a of $U(N)$ are normalized by $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ where the index a runs from 1 to N^2. The problem of calculating the cross-section is now straightforward, and can be found in any textbook on quantum field theory, see for example [28], pg. 107. The field A^a_α will have two physical polarizations for each a. At leading order in $\lambda = g_{YM}^2N$, each will couple to an in-coming dilaton ϕ (solid line below) via the following Feynman diagram



$$= \quad \mathcal{M} = -\frac{2\widetilde{\kappa}_{10}}{\sqrt{2}}{p_1}^\alpha p_{2\alpha}$$

where the ${p_i}^\alpha$ are the four-momenta of the final state photons (wiggly lines above). The cross-section is then given by

$$\sigma = \frac{1}{2}\frac{1}{2\omega}\int\frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\int\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}(2\pi)^4\delta(E_1+E_2-\omega)\delta^3(\vec{p}_1+\vec{p}_2)\,|\mathcal{M}|^2 = \frac{\widetilde{\kappa}_{10}^2\omega^3}{64\pi} \tag{1.85}$$

[14]The in-coming dilaton s-wave cannot be converted into a pair of fermions on the brane because the coupling, involving the kinetic term $\bar{\psi}\partial\!\!\!/\psi$, gives an odd power of the momentum.

where ω is the energy of the in-coming dilaton and the leading factor of $1/2$ accounts for the fact that the final-state photons are identical. Since we have $2N^2$ species of photons, where the 2 counts the number of physically distinct polarizations, we have

$$\sigma_{tot} = \frac{N^2 \widetilde{\kappa}_{10}^2 \omega^3}{32\pi}. \tag{1.86}$$

Using the fact that $\widetilde{\kappa}_{10}^2 = (2\pi)^7 g_s^2 \alpha'^4/2$, and (1.65) we see that this is identical to (1.80). Therefore, for the dilaton s-wave, the stack of D3-branes in flat 10-dimensional space absorbs exactly the same as the throat of the geometry (1.61). In fact the agreement is suspicious. The supergravity result (1.80) is valid in the supergravity approximation, i.e. when $R^4/\alpha'^2 = g_{YM}^2 N \gg 1$, but we performed our D-brane calculation only to leading order in $\lambda = g_{YM}^2 N$, i.e. in the $R^4/\alpha'^2 \ll 1$ limit. In fact the higher order corrections to (1.86) vanish as a result of a non-renormalization theorem [11]. The cross-sections for other closed string modes, namely fermions and gravitons, were found to agree similarly in [10].

This result was an indication that there was a duality emerging between strongly coupled super-Yang-Mills and IIB strings on a weakly curved background. In fact there was already an indication from a study of entropy [12], that a correspondence may be at play. It was found that the entropy of the weakly coupled super-Yang-Mills theory at small temperature agreed with that of the corresponding near-extremal black hole entropy, up to a factor of 4/3. This near match for quantities at opposite ends of the coupling spectrum was notable. Maldacena [13] eventually codified these observations into a conjecture which has largely shaped string theory research in the interim.

1.5.3 The Maldacena conjecture

The central theme in the Maldacena conjecture is a *decoupling limit*. Specifically, Maldacena considered taking $R, \alpha' \to 0$ while keeping $R^4/\alpha'^2 = \lambda = 4\pi g_s N$ fixed. This limit sends the cross-section (1.86) to zero; the strings in the throat of (1.61) (on the stack of D-branes) are decoupled from those in the asymptotic region (away from the stack of D-branes). This results in two pictures, each containing two decoupled theories. In the throat-geometry picture we have closed, type IIB strings propagating in the $r \ll R$ region of (1.61)

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dx_i^2\right) + \frac{R^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right) \tag{1.87}$$

which are decoupled from closed IIB strings propagating out at $r \gg R$, which is just 10-dimensional flat space. In the D-brane picture we have $\mathcal{N} = 4$ super-Yang-Mills in four spacetime dimensions with gauge group $U(N)$ on the stack of D-branes, while decoupled closed, type IIB strings propagate in the bulk spacetime which is 10-dimensional flat space. The two pictures share a decoupled theory: closed type IIB strings in 10-dimensional flat space. The Maldacena conjecture [13] posits that the other two decoupled theories are also equivalent, that is

$\mathcal{N} = 4$ *super-Yang-Mills in four spacetime dimensions with gauge group* $U(N)$ *is dual to type IIB string theory on the background* $AdS_5 \times S^5$

where we have identified (1.87) as the metric of five dimensional anti de-Sitter space times a five-sphere. We should also not forget that this background includes the form-field strength (1.62). What does this duality imply about the two theories? The super-Yang-Mills has two parameters; N the rank of the gauge group, and $\lambda = g_{YM}^2 N$ the 't Hooft coupling. The decoupling limit keeps λ fixed. This means that when N is varied, $g_{YM}^2 = 4\pi g_s$ varies inversely. Thus the closed string coupling varies inversely with N, so that when N is large the closed string dynamics are captured by their tree-level or classical limit, see figure 1.13. The 't Hooft coupling itself may also be varied. From (1.65) we see that λ measures the ratio of the AdS (and five-sphere) curvature to the string length ($\sqrt{\alpha'}$). Thus when λ is large, i.e. the gauge theory is strongly coupled, the "stringiness" of the closed strings is suppressed and they are well approximated by their low-energy point-particle limit: type IIB supergravity. The beauty of this correspondence is that (at least for large N) it is

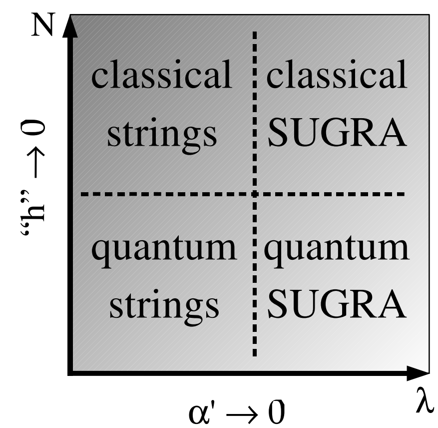


Figure 1.13: The parametric limits of the AdS/CFT correspondence. The rank of the super-Yang-Mills gauge group N controls quantum corrections in the closed string theory; these correspond to non-planar gauge theory processes. The 't Hooft coupling controls the ratio of the closed string background curvature to the string length; large 't Hooft coupling corresponds to point-particle supergravity ("SUGRA"). Here "h" refers to the effective Planck's constant - i.e. the parameter controlling "quantumness".

precisely where the analytical techniques fail in the gauge theory, i.e. at large λ, that the dual string theory simplifies to classical supergravity on a weakly curved background. We have analytical control over the dual theory in this regime, and so we have finally realized a string description of a strongly coupled gauge theory that will allow analytical calculations.

1.5.4 Preliminary evidence for AdS/CFT: symmetries

We have presented a conjecture in section 1.5.3 with no evidence or proof. What is the main motivation, above and beyond the cross-section calculations of section 1.5.2, for suggesting this duality? The answer is that the two theories share the same symmetry groups. $\mathcal{N} = 4$,

$d = 4$ supersymmetric Yang-Mills theory is believed to be a *conformal field theory* or CFT. The meaning of this is that the full interacting, quantum mechanical theory is invariant under conformal transformations. These are angle-preserving transformations which include global rescalings

$$x^\mu \to \Lambda x^\mu \tag{1.88}$$

and the so-called *special conformal transformations*

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2a_\mu x^\mu + a^2 x^2} \tag{1.89}$$

where a^μ is some real d-vector. A natural consequence of invariance under (1.88) is that the β-function for the coupling g_{YM} is identically zero. This means that the coupling in a CFT does not run at all - it is a free parameter in the theory. This is important for the duality of AdS/CFT since we would like to be able to vary the 't Hooft coupling λ freely (see figure 1.13). The conformal group extends the symmetry of flat space (1.31), i.e. the Poincaré group, via the inclusion of the generators of (1.88) D, and those of (1.89) K_μ. The extra algebraic relations are

$$\begin{gathered}[D, P_\mu] = -iP_\mu, \qquad [P_\mu, K_\nu] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D, \qquad [D, K_\mu] = iK_\mu, \\ [M_{\mu\nu}, D] = 0, \qquad [M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu).\end{gathered} \tag{1.90}$$

In fact the conformal group is isomorphic to $SO(2, d)$. This can be seen via the following assignments

$$J_{\mu\nu} = M_{\mu\nu}, \qquad J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu), \qquad J_{\mu\,(d+1)} = \frac{1}{2}(K_\mu + P_\mu), \qquad J_{(d+1)\,d} = D, \tag{1.91}$$

for which $[J_{MN}, J_{IJ}] = i\eta_{NI}J_{MJ} - i\eta_{MI}J_{NJ} - i\eta_{NJ}J_{MI} + i\eta_{MJ}J_{NI}$ simply gives the conformal group relations, where $I, J, M, N = 0, \ldots, d+1$ with signature $(-, +, \ldots, +, -)$. $\mathcal{N} = 4$ super-Yang-Mills (1.69) contains another important symmetry group, called R-symmetry. This symmetry rotates the six scalar fields Φ^I into one another; therefore this group is $SO(6)$. On the level of bosonic symmetries, this is it. Thus the bosonic symmetries of $\mathcal{N} = 4$ super-Yang-Mills is $SO(2, 4) \times SO(6)$.

Anti de-Sitter space may be defined as the embedding of a hyperboloid in a two-time signature space

$$ds^2 = -dx_0^2 - dx_{d+1}^2 + (dx_i)^2, \qquad i = 1, \ldots, d. \tag{1.92}$$

The hyperboloid is embedded as

$$x_0^2 + x_{d+1}^2 - (x_i)^2 = R^2 \tag{1.93}$$

which is manifestly $SO(2, d)$ invariant. It is then plain to see that the isometry group of the space $AdS_5 \times S^5$ is $SO(2, 4) \times SO(6)$. We have therefore an exact matching of the bosonic symmetries between $\mathcal{N} = 4$ super-Yang-Mills and fields on $AdS_5 \times S^5$. In fact it can be shown that the full supergroup of type IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills is $SU(2,2|4)$, of which $SO(2,4) \times SO(6)$ is the bosonic subgroup.

It will be useful to explore anti de-Sitter space more thoroughly. The hyperboloid (1.93) may be coordinatized using the following relations

$$x_0 = R\cosh\rho\cos\tau, \qquad x_{d+1} = R\cosh\rho\sin\tau, \qquad x_i = R\sinh\rho\,\Omega_i, \qquad (\Omega_i)^2 = 1 \tag{1.94}$$

where Ω_i are an embedding of the $(d-1)$-sphere. This leads to so-called *global AdS*

$$ds^2 = R^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2) \tag{1.95}$$

where the coordinate τ is unwrapped from its fundamental domain $[-\pi, \pi]$ to $[-\infty, \infty]$ in order to avoid closed timelike curves, while $\rho \in [0, \infty]$. Another important coordinatization is

$$\begin{gathered} x_0 = \frac{1}{2r}\left(1 + r^2(R^2 + \vec{z}^2 - t^2)\right), \qquad x_d = \frac{1}{2r}\left(1 - r^2(R^2 - \vec{z}^2 + t^2)\right), \\ x_a = Rrz_a, \qquad x_{d+1} = Rrt, \qquad a = 1, \ldots, d-1 \end{gathered} \tag{1.96}$$

where $r \geq 0$ and t and $\vec{z}$ are unconstrained. This gives the *Poincaré patch*

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{z}^2\right) + \frac{R^2}{r^2}dr^2 \tag{1.97}$$

which we saw in (1.87). The global coordinates (1.95) cover the entire hyperboloid (1.93) once, the Poincaré patch, as we will see, covers only half of it. The relationship between the global and Poincaré coordinates systems is most easily seen when $d = 2$, i.e. for AdS_3. In this case the Ω_i parametrizes a circle whose parameter we will take as ϕ, and we have

$$\begin{gathered} r = \frac{1}{R}\left(\cosh\rho\cos\tau - \sinh\rho\cos\phi\right), \\ t = \frac{R\cosh\rho\sin\tau}{\cosh\rho\cos\tau - \sinh\rho\cos\phi}, \qquad z = \frac{R\sinh\rho\sin\phi}{\cosh\rho\cos\tau - \sinh\rho\cos\phi}. \end{gathered} \tag{1.98}$$

Since $r \geq 0$, we must have $\cos\tau \geq \tanh\rho\cos\phi$. The boundary of the patch is therefore given by the following curves (see figure 1.14, right panel, where ρ is set to infinity)

$$\tau = \pm\arccos\left(\tanh\rho\cos\phi\right). \tag{1.99}$$

One may verify that the area (in the τ-ϕ plane) of the patch is therefore given by

$$4\int_0^{\pi} d\phi \arccos\left(\tanh\rho\cos\phi\right) = 2\pi^2 \tag{1.100}$$

independently of ρ. This is one half of the total area, $4\pi^2$ (since τ and ϕ are $\in [0, 2\pi]$).

Anti de-Sitter space also has a time-like boundary which is most easily seen through the change of coordinates $\tan\theta = \sinh\rho$, where $\theta \in [0, \pi/2]$. The global AdS metric (1.95) then takes the following form

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_i^2). \tag{1.101}$$

The boundary is found at spatial infinity ($\theta = \pi/2$ or $\rho = \infty$) and is of the form $\mathbb{R} \times S^{d-1}$. A Penrose diagram of the space is shown in figure (1.14). The boundary is time-like and

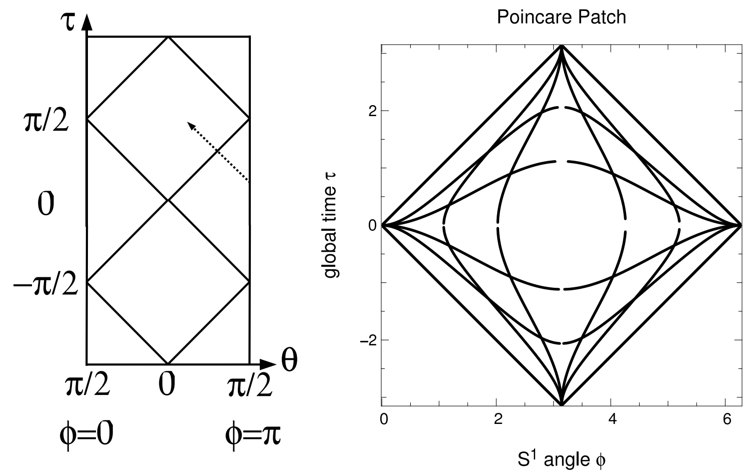


Figure 1.14: A Penrose diagram of anti de-Sitter space (1.101) is shown on the left. The diamonds are the paths of light-rays beginning at $\theta = 0$, $\tau = -\pi$ and reflecting off the boundary at $\theta = \pi/2$. A signal from the boundary (which is at spatial infinity) may propagate into the spacetime in finite coordinate time. Note that the diagram should be understood to be a fundamental domain which is periodically continued to $\tau = [-\infty, \infty]$ The angle ϕ is understood as the azimuthal angle in the Ω_i portion of the metric. On the right the boundary ($r = \infty$) of the Poincaré patch (1.97) is displayed for $d = 2$. The boundary of global AdS_3 may be envisioned as a cylinder with a coordinate τ running along the length of the cylinder and the angle ϕ going around it. The Poincaré patch fits into half a fundamental domain, bounded by null surfaces at spatial infinity ($z = \pm\infty$) forming a diamond shape which wraps around the cylinder. The horizontal curves are lines of constant z, while the vertical curves are lines of constant t.

it takes a finite amount of coordinate time for a signal to propagate from the boundary at spatial infinity to any point in the space. This implies that information may be gained from or lost to the boundary, and in this respect anti de-Sitter space is very similar to Minkowski space in a box. This will be important for us in the next section. The boundary in the Poincaré patch is at $r = \infty$ and is given by $ds^2 = r^2(-dt^2 + d\vec{z}^2)/R^2$; i.e. it is a conformal rescaling of flat d-dimensional space. In this sense the Poincaré coordinates "lose the point at infinity" required to restore the $\mathbb{R} \times S^{d-1}$ topology of the boundary.

1.5.5 The field-operator dictionary and the GKP-W relation

The AdS/CFT correspondence alleges an equivalence between a conformal field theory and a string theory on $AdS_5 \times S^5$, but how is the equivalence seen? In order to specify what is equivalent to what, a dictionary is required which translates a problem posed in one setting into the language of the other, and vice-versa. Such a dictionary is given by the GKP-W relation [29, 30], named for Gubser, Klebanov, Polyakov, and Witten. The first question we should ask is what are the meaningful (physical) quantities in each of the theories that are eligible for comparison. A conformal field theory or CFT has no scale, it is therefore meaningless to discuss asymptotically free wave-packets, and this precludes an S-matrix and the concept of particles with definite mass. An important class of invariants that a CFT does possess is the scaling dimensions of operators. These must relate to some other invariant in the gravity (string) theory. Supergravity in $AdS_5 \times S^5$ does have a scale and asymptotic mass eigenstates. We mentioned in section 1.5.4 that in some respects AdS space is similar to Minkowski space in a box. In fact, a unique solution to the Laplace equation for a (for example scalar) field $\phi(r; \vec{z}, t)$ on AdS requires the specification of boundary data $\phi_0(\vec{z}, t) = \phi(\infty; \vec{z}, t)$. Keeping these facts in mind, the absorption cross-section calculations of section 1.5.2 hint at what the relation between the CFT and the supergravity should be. Recall that the coupling of the dilaton to the D-brane worldvolume theory (at low energy) was (1.83) $\sim \mathrm{Tr} \int d^4x \phi_0 F^2$. That is, the bulk closed string mode described by the field ϕ, interacts with the worldvolume theory via a local (i.e. proportional to the value of ϕ on the space where the worldvolume theory lives, that is, ϕ_0) coupling to an operator of the CFT. In the throat of the geometry, ϕ represents a minimization of the supergravity action on AdS_5, subject to the boundary condition ϕ_0. This minimized action therefore appears equivalent to the addition of the operator $\mathcal{O} = \mathrm{Tr}\, F^2$ to the worldvolume theory's action, $S_{WV} \to S_{WV} + \int d^4x \phi_0 \mathcal{O}$. To be more precise, the GKP-W relation is

$$\boxed{\left\langle \exp\left(\int d^4x\, \phi_0\, \mathcal{O}\right)\right\rangle_{CFT} = \exp\left(-S_{\mathrm{SUGRA}}(\phi)\right)}. \tag{1.102}$$

It may seem confusing that ϕ_0 is at once the value of the field ϕ on the stack of D-branes and the value on the boundary of AdS_5. What must be remembered is that in the decoupling limit (see section 1.5.3), the throat region, which is identified with the position of the D-branes, is blown-up to the entire space $AdS_5 \times S^5$. In this *near horizon* geometry, where has the corresponding position of the stack of D-branes gone? The answer is to the boundary of AdS_5. Indeed we saw in section 1.5.4 that this boundary is conformally equivalent to four-dimensional flat space. In fact, the $SO(2,4)$ isometry group of AdS_5 acts upon this boundary as the four-dimensional conformal group. It should be emphasized that it is a mistake to think of the CFT as living on the boundary of AdS_5 *simultaneously* with the supergravity in the bulk. They are conjectured as equivalent descriptions of the same physics; we can either work with the full AdS_5 supergravity or we can throw that away and answer the same questions with the holographic CFT. To see that this is so we can work out a simple example in which it will be revealed that the relation (1.102) actually implies an equivalence between the scaling dimension of the operator $\mathcal{O}$ and the mass of the associated field ϕ. It is simplest to employ the Poincaré metric (1.97), in Euclidean signature, and with the coordinate redefinition $r \to R^2/y$,

$$ds^2 = \frac{R^2}{y^2}(dy^2 + dx_i^2) \tag{1.103}$$

where the Euclideanized boundary space at $y = 0$ is now covered by the coordinates x_i. The Green's function corresponding to a field $\phi(y, x_i)$ specified by boundary data $\phi(0, x_i)$ is most easily obtained from an $SO(1, d+1)$ transformed version depending only on y [30]

$$K(y) = Cy^d \tag{1.104}$$

where C is some constant. This ansätz obeys the equation of motion $\Box\phi = 0$ and the boundary condition $K(\infty) = \infty$. Under the $SO(1, d+1)$ inversion

$$y \to \frac{y}{y^2 + x_i^2}, \qquad x_i \to \frac{x_i}{y^2 + x_i^2} \tag{1.105}$$

the Green's function (1.104) becomes

$$K(y, x_i) = C\frac{y^d}{(y^2 + x_i^2)^d} \tag{1.106}$$

which gives (for properly chosen C) the behaviour $K(0, x_i) = \delta^d(x_i)$ at the boundary point $y = 0$. This is the desired boundary condition for a *bulk-to-boundary propagator*, as it defines the bulk field $\phi(y, x_i)$ in terms of its boundary value $\phi(0, x_i)$ in the following manner

$$\phi(y, x_i) = \int d^d\widetilde{x}_i \frac{Cy^d}{(y^2 + (x_i - \widetilde{x}_i)^2)^d}\, \phi(0, \widetilde{x}_i). \tag{1.107}$$

We should now plug this solution into the RHS of (1.102). The action for a massless scalar field is

$$S[\phi] = \frac{1}{2}\int d^d x_i \int dy\, \sqrt{g}\, \partial_\mu\phi\, \partial^\mu\phi \tag{1.108}$$

where μ runs over all $d+1$ coordinates of (1.103) which is also the metric $g_{\mu\nu}$ refers to. Plugging the solution (1.107) into this action gives only a surface term at $y = 0$, since the main part of the action vanishes by the equation of motion. It is straightforward to show that

$$S[\phi(y, x_i)] = \frac{Cd}{2}\int d^d x_i \int d^d\widetilde{x}_i \frac{\phi(0, x_i)\, \phi(0, \widetilde{x}_i)}{(x_i - \widetilde{x}_i)^{2d}}. \tag{1.109}$$

The two-point function of an operator $\mathcal{O}(x)$ in the CFT of conformal weight d has the following behaviour

$$\langle \mathcal{O}(x_i)\, \mathcal{O}(\widetilde{x}_i)\rangle = \frac{\mathcal{C}}{(x_i - \widetilde{x}_i)^{2d}} \tag{1.110}$$

where $\mathcal{C}$ is a constant. It is immediately seen that for the appropriate choice of the constant C, the LHS of (1.102) is exactly (1.109), where we have taken only the quadratic term in the expansion of the exponentials, i.e. we are comparing two-point functions. This whole story is repeated for the case of a massive field ($(\Box - m^2)\phi = 0$) with the replacement[15]

$$d \to \Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right). \tag{1.111}$$

We therefore have that the dual of the scaling dimension Δ of an operator $\mathcal{O}$ in the CFT is related to the mass m of the dual AdS field by (1.111).

1.5.6 Beyond two-point functions

The AdS/CFT correspondence has passed many tests beyond the two-point function presented in section 1.5.5. We will not give a detailed account of the various successes of the correspondence here, as that would fill several review papers. Three point functions are well understood [32], and are protected by conformal invariance in the CFT. Four point functions do not share this protection, but have been studied extensively (see references [92] through [104]). There are also large bodies of work concerning Wilson loops (see chapter 4), M-theory (see chapter 3), thermodynamics, D-brane states, macroscopic strings, viscosity, black-hole entropy and information loss, and more. Indeed Maldacena's original paper [13] has been cited over 4500 times at the time of writing, and appears to be increasing roughly linearly with time, see figure 1.15. The AdS/CFT correspondence has been the main focus of string

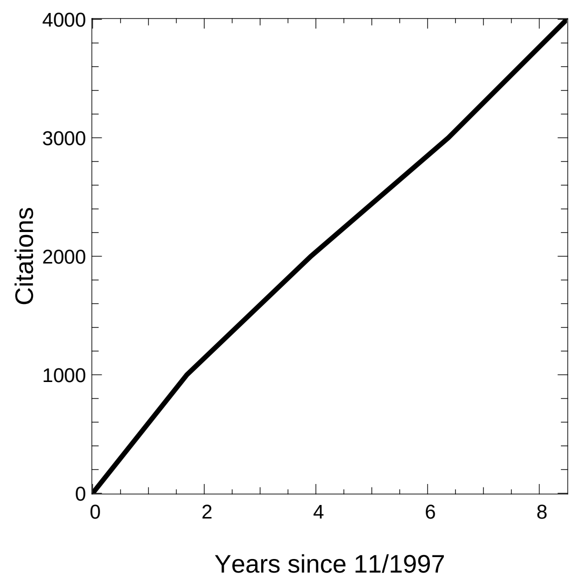


Figure 1.15: Citations of Maldacena's original paper [13] as a function of time, loosely interpreted from the citebase website [105]. The current SPIRES count is more than 4500.

theory research for the past decade. Although a proof of the correspondence is still lacking,

[15]The number of CFT spacetime dimensions remain d. The other difference is in the relation of the CFT scalar $\phi_0(x_i)$ to the boundary behaviour of the bulk field $\phi(y, x_i)$; the general relationship is $\lim_{y\to 0}\phi(y, x_i) = y^{d-\Delta}\phi_0(x_i)$.

no test of AdS/CFT has returned a negative result. Much of the work on the correspondence has naturally focused on the classical regime where large-N super-Yang-Mills at strong coupling is compared with supergravity. True quantum tests, involving $1/N$ corrections, are less prevalent and less definitive, as we will see in chapter 2. The quantum and stringy limits of AdS/CFT are the most important avenues to study, in order to further the evidence that the correspondence is indeed correct.

1.6 Summary

We began this chapter with the clash of two titans, strongly coupled gauge theory and gravity. String theory, devised to study the former, was retired and then refitted to study the latter. In the end it led to a remarkable observation, that in a certain context, the two titans are reflections of each other, descriptions of the same physics encoded in a different language. Further, the questions that are most difficult to answer in one description are the easiest to answer in the other. The stage is thus set for an exploration of this gauge/string duality. In the following three chapters of this thesis, we will present work concerning the quantum (chapter 2), thermodynamic (chapter 3), and classical (chapter 4) aspects of the AdS/CFT correspondence. This research represents some important steps in the very long journey to fully understanding the breadth and mechanisms of this remarkable duality.

Chapter 2

Light-cone string field theory on the plane-wave

> I call the loose snow to fly
> from your rooftops, the timorous doves
> from your rafters' sanctum.
> — Ruth Taylor *The Dragon Papers*, Emanations

String theory, despite all its complexity and scope, remains a first-quantized theory. It is fundamentally the quantum dynamics of a single relativistic string. The theory lacks a Lagrangian which would dictate the full interacting theory of a field of strings. From the point particle analogue point of view, we have only quantum mechanics - not quantum field theory. This lack of completeness has not prevented string theorists from considering on-shell string interactions in some detail (c.f. [33–36]), guided simply by the symmetries inherent in the first quantized theory, and by the manner in which strings interact with background fields. To be more precise, we have some control over interactions in the supergravity limit (as discussed in section 1.4.6), where we know the full point-particle Lagrangian is given by supergravity. On-shell scattering amplitudes for low-lying string modes (i.e. few excitations), with small numbers of loops (< 3), were calculated very early on. Continuing string interactions off the mass-shell has enjoyed a certain amount of success using Witten's cubic string field theory [37], see [38] for a recent review. From the point of view of light-cone gauge quantization (see section 1.4.5), the description of the most basic interaction - the three string vertex - is simply a map between the collections of oscillators (1.45) which completely specify each string, see figure 2.1.

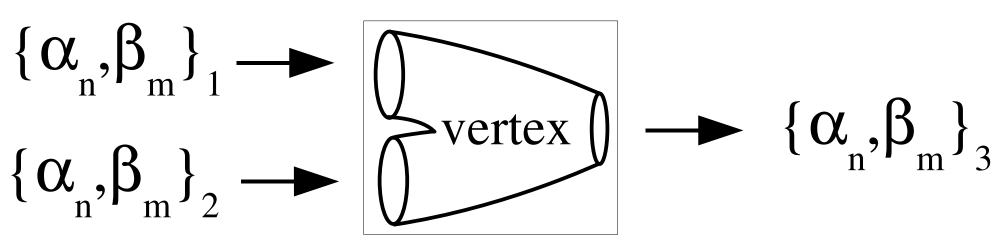


Figure 2.1: In light-cone string field theory the cubic interaction vertex may be considered as a map between the collections of oscillators α_n and β_m which completely specify each string.

Such a map was developed by Green, Schwarz, and Brink [45, 46] in the early 1980's, for type

II superstrings in a flat target space background. In 2002, Spradlin and Volovich [53, 54] generalized this work to the plane-wave background. In that context, the light-cone string field theory may be used to calculate one-loop shifts to the masses of string states. These are dual to anomalous dimensions of certain operators in $\mathcal{N} = 4$ super-Yang-Mills (in a given limit) and may be compared as a check on the AdS/CFT correspondence[1]. This chapter will introduce this program, and present original work by the author which clarified some oblique issues concerning the importance of intermediate string states in the one-loop mass shifts, proved the finiteness of those shifts, and culminated in the best match so far discovered between the gauge and string theory results using a version of the string interaction vertex proposed by Dobashi and Yoneya [57].

2.1 The plane-wave background and the BMN limit of $\mathcal{N} = 4$ SYM

In section 1.5 the AdS/CFT correspondence was introduced as a conjectured equivalency between type IIB superstrings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM. An early stumbling block encountered by investigators of the correspondence was solving the string sigma model (curved space analogue of 1.39) on this background. This remains an obstacle which has yet to be overcome. Without the rudimentary information provided by the free string spectrum on $AdS_5 \times S^5$, understanding or even testing the AdS/CFT correspondence beyond the supergravity limit is extremely limited. In 2001, a new type IIB background space was found [58] which, like $AdS_5 \times S^5$, is maximally supersymmetric. This is the so-called *plane-wave* background and is given by

$$ds^2 = -2\,dx^-dx^+ - \mu^2(x_i)^2(dx^+)^2 + dx_i^2, \qquad F_{+1234} = F_{+5678} = \mu \times \text{const.} \tag{2.1}$$

where $+$ and $-$ denote light-cone directions, $i = 1, \ldots, 8$, and μ is a real, positive constant. This space may be obtained via a Penrose limit [62] of $AdS_5 \times S^5$ where the neighbourhood of a null geodesic (an equator of the five-sphere) is "zoomed-in" upon. To see this consider global $AdS_5 \times S^5$, given by the following metric

$$ds^2 = R^2\left(-\cosh^2\rho\,dt^2 + d\rho^2 + \sinh^2\rho\,d\Omega_3^2 + d\theta^2 + \cos^2\theta\,d\psi^2 + \sin^2\theta\,d\Omega'^2_3\right). \tag{2.2}$$

The geodesic whose neighbourhood will be magnified is given by $\rho = \theta = 0$, i.e. the equator parametrized by the angle ψ. To realize this magnification we define $\hat{x}^+ = \frac{1}{2}(t+\psi)$, $\hat{x}^- = (t-\psi)$, then re-scale the coordinates as follows

$$\hat{x}^+ \to x^+ = \frac{1}{\mu}\hat{x}^+, \qquad \hat{x}^- \to x^- = \mu R^2 \hat{x}^-, \qquad \rho = \frac{r}{R}, \qquad \theta = \frac{y}{R}, \qquad R \to \infty \tag{2.3}$$

it is then easy to see that the metric (2.1) is obtained. It was not long before the κ-symmetric Green-Schwarz superstring action was found and the string equations of motion were solved

[1] For reviews of the plane-wave string / gauge theory duality see [51] and [52].

on this simplified background [59, 60]. The free string spectrum, given in terms of the light-cone Hamiltonian, is as follows

$$H_{\text{l.c.}} = p^- = \frac{1}{\alpha' p^+} \sum_n N_n \sqrt{n^2 + (\mu\alpha' p^+)^2} \tag{2.4}$$

where N_n denotes the occupation number (number operator for fermionic and bosonic oscillators), and positive n denotes left-moving modes while $n < 0$ denotes right-moving modes.

Because the plane-wave was obtained from $AdS_5 \times S^5$ through a continuous scaling procedure, we may use the AdS/CFT correspondence to provide a translation of the quantities found here to analogous ones in $\mathcal{N} = 4$ SYM. We saw in section 1.5.4 that the R-symmetry of SYM is the analogue of the $SO(6)$ symmetry group of the five-sphere in $AdS_5 \times S^5$. We thus expect the (appropriate) R-charge J of operators in the SYM to be dual to the angular momentum $-i\partial_\psi$ of string states about the five-sphere. The energy of a string state $i\partial_t$ should be dual to the conformal dimension Δ of those operators. We therefore have

$$\begin{aligned} p^- &= i\partial_{x^+} = i\mu\, \partial_{\hat{x}^+} = i\mu(\partial_t + \partial_\psi) = \mu(\Delta - J) \\ p^+ &= i\partial_{x^-} = \frac{i}{\mu R^2}\, \partial_{\hat{x}^-} = \frac{i}{\mu R^2}\frac{1}{2}(\partial_t - \partial_\psi) = \frac{\Delta + J}{2\mu\sqrt{g_{YM}^2 N}\,\alpha'} \end{aligned} \tag{2.5}$$

where we have used (1.65) and (1.70). Since we are taking the limit $R \to \infty$, states with finite p^+ must have $J \sim R^2 \sim \sqrt{g_{YM}^2 N}$. Further, finite p^- then implies $\Delta \sim J$. This allows us to rewrite the string spectrum (2.4) in terms of gauge theory quantities, since according to these scalings $\mu\alpha' p^+ = J/\sqrt{\lambda}$ where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. We therefore have

$$\frac{1}{\mu} p^- = \sum_n N_n \sqrt{1 + \frac{n^2 \lambda}{J^2}}. \tag{2.6}$$

The specific operators which are dual to free strings were identified in a milestone paper by Berenstein, Maldacena, and Nastase (BMN) in 2002 [61]. The single string vacuum is labelled by its light-cone momentum p^+, which is a free parameter in the theory. This corresponds to a state with $p^- = 0$, i.e. an operator with $\Delta = J$, specifically the BMN operator in this case is

$$\mathcal{O}_J = \frac{1}{\sqrt{J N^J}} \operatorname{Tr} Z^J \leftrightarrow |0; p^+\rangle \tag{2.7}$$

where $Z = \Phi^5 + i\Phi^6$, corresponds to the plane in $\mathbb{R}^6$ (i.e. the 5-6 plane) which the five-sphere equator parametrized by ψ sits in. It is clear that the R-charge (corresponding to the 5-6 plane) of this operator is J, as it contains J factors of the field Z. Further, one may verify the conformal dimension at zero Yang-Mills coupling (equivalently zero string coupling) is precisely J in the large N (planar) limit, i.e.

$$\langle \mathcal{O}_J(x)\, \mathcal{O}_{J'}(0) \rangle = \left(\frac{g_{YM}^2}{4\pi} \right)^J \frac{\delta_{JJ'}}{(x^2)^J}. \tag{2.8}$$

The BMN operators corresponding to excited string states were also identified in [61]. The first excited states of the string are those obtained by acting on the vacuum with two oscillators, such that the total worldsheet momentum vanishes (as required by level matching (1.29)). The reflection of string oscillators on the gauge theory side of the correspondence are the insertion into the operator $\mathcal{O}_J$ fields dubbed *impurities*. We will encounter the full treatment of the string theory in the next section; for now it suffices to say that there are 8 transverse bosonic oscillators α^i_n (labelled by the spacetime index $i = 1, \ldots, 8$) and 8 fermionic super-counterparts β^a_n (labelled by an $SO(8)$ spinor index a). These oscillators are in one-to-one correspondence with the fields of the gauge theory in the following way

$$\begin{aligned} &\alpha^{\dagger j}_n \to D_j Z \qquad j = 1, \ldots, 4 \\ &\alpha^{\dagger k}_n \to \Phi^{k-4} \qquad k = 5, \ldots, 8 \\ &\beta^{\dagger a}_n \to \chi^a \end{aligned} \tag{2.9}$$

where D_j is the gauge-covariant derivative and χ^a are the fermionic fields of $\mathcal{N} = 4$ SYM. These impurities are interleaved into the trace of (2.7) by adding a position dependent phase to each. Even though a single impurity state is unphysical, we present it here as an instruction

$$\frac{1}{\sqrt{JN^{J+1}}} \sum_{l=0}^{J-1} e^{\frac{2\pi i n l}{J}} \operatorname{Tr} Z^l \Phi^1 Z^{J-l} \leftrightarrow \alpha^{\dagger 5}_n |0; p^+\rangle. \tag{2.10}$$

Note that by cyclicity all of the traces in the sum are equivalent, leading to an overall factor of $\sum_{l=0}^{J-1} \exp(2\pi i n l/J) = 0$, since $n \in \mathbb{Z}$. The unphysical nature of the state thus takes care of itself by being identically zero. Note that this operator has $J+1$ fields, while its 5-6 plane R-charge remains J. Therefore $\Delta - J = 1$, corresponding to one unit of light-cone energy. A two-impurity state is built in the same way, by simply adding a second impurity with its own phase factor and summing over positions of insertion in the original chain of J Z's,

$$\sum_{0 \le k \le l \le J-1} \left[\operatorname{Tr} Z^k \Phi^1 Z^{l-k} \Phi^2 Z^{J-l} e^{\frac{2\pi i n k}{J}} e^{-\frac{2\pi i n l}{J}} + \operatorname{Tr} Z^k \Phi^2 Z^{l-k} \Phi^1 Z^{J-l} e^{\frac{2\pi i n l}{J}} e^{-\frac{2\pi i n k}{J}} \right] \tag{2.11}$$

where we have dropped the normalization. Using the cyclicity of the trace, this expression is simplified to the compact form

$$\alpha^{\dagger 5}_n \alpha^{\dagger 6}_{-n} |0; p^+\rangle \leftrightarrow \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} e^{\frac{2\pi i n l}{J}} \operatorname{Tr} \Phi^1 Z^l \Phi^2 Z^{J-l} = \mathcal{O}^{12}_J \tag{2.12}$$

where the normalization has been restored and the dual string state indicated. This procedure may be generalized to include any number of impurities, c.f. [68].

We now have a picture of perturbative strings as operators which consist of a very long string of $J \sim R^2$ fields, with a few impurities sprinkled along it. We also know that the 't Hooft coupling $\lambda \sim R^4$ is taken to infinity. It would not be surprising to find that a new coupling $\lambda' \equiv \lambda/J^2$ might arise in the interactions between the BMN operators, since it is tunably small and serves as the perturbative parameter in the expansion of the free string energy (2.6). We have found that in the zero 't Hooft coupling limit, $\Delta - J$ for the BMN operators are simply integers, corresponding to (2.6) with $\lambda = 0$. If we turn this coupling on, we expect to reproduce the entire Taylor expansion of the square root. Indeed this has been verified to a few orders in perturbation theory [67–73], and to all orders via a superspace formalism proof [66]. The leading contribution is derived from the quartic scalar interaction in the action for $\mathcal{N} = 4$ SYM (1.69)

$$V = -4g_{YM}^2\left(\mathrm{Tr}\left|[Z,\Phi^1]\right|^2 + \mathrm{Tr}\left|[Z,\Phi^2]\right|^2\right) \tag{2.13}$$

where we have shown the terms which will be important for the operator (2.12). The leading correction to the two-point function $\langle \mathcal{O}_J^{12}(x)\,\mathcal{O}_J^{12\,\dagger}(0)\rangle$ is depicted in figure 2.2. The interaction (2.13) connects the term in $\mathcal{O}(x)$ with that of $\mathcal{O}^\dagger(0)$ in which the impurity Φ is moved along by one Z field. This gives the leading contribution to the anomalous dimension

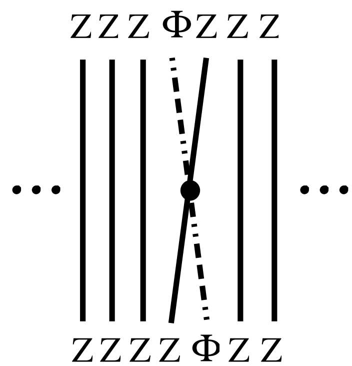


Figure 2.2: The leading correction to the scaling dimension of the operator (2.12) is given by the quartic vertex which connects the terms of $\mathcal{O}(x)$ and $\mathcal{O}^\dagger(0)$ in which one of the impurities is shifted by one unit.

as follows

$$\langle \mathcal{O}_J^{12}(x)\,\mathcal{O}_J^{12\,\dagger}(0)\rangle \sim \frac{1}{(x^2)^{J+2+\gamma}}, \qquad \gamma = n^2\lambda' + \ldots \tag{2.14}$$

thus reproducing the leading term in the expansion of (2.6) for $N_n = 2$, i.e. $\Delta - J = 2 + n^2\lambda'$.

As was discussed in section 1.3, string loop diagrams are reflected in the dual gauge theory by non-planar diagrams, which are suppressed by powers of $1/N$ compared to the planar diagrams. In fact in the BMN limit, the string-loop counting parameter turns out to be $J^2/N \equiv g_2$. In order to see this one must consider the non-planar contributions to the two-point function of BMN operators. Already in the free-field limit, when $\lambda' = 0$, the coupling g_2 emerges readily. Consider the genus-1 contribution to the two-point function of $\mathcal{O}_J$ given in (2.7). There are two classes of diagrams which can be drawn on a torus without crossing lines, which cannot be drawn on a sphere. They correspond to splitting the J propagators into either 3 or 4 groups as shown in figure 2.3. There are therefore

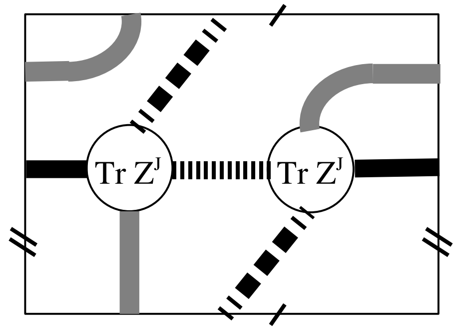


Figure 2.3: The torus diagram for the two-point function of the operator (2.7) in the free-field limit is given by separating the J propagators into four groups as shown above. The group depicted by the grey line may be removed, leaving a contribution from separation into three groups. These diagrams may not be drawn on the sphere without crossing lines. The J fields of each operator have been arranged in a circle to reflect the cyclicity of the trace.

$\binom{J}{3} + \binom{J}{4} \simeq J^4/4!$ ways of contracting the fields, and each is suppressed by $1/N^2$ compared to its planar counterpart. Therefore the quantity $J^4/N^2 = g_2^2$ emerges naturally. When the coupling λ' is turned on, torus (and higher genus) diagrams will contribute to the process shown in figure 2.2, for example. This leads to g_2 terms in the anomalous dimension of the BMN operators. Figure 2.4 shows one of the torus diagrams which contributes to the leading g_2 contribution to γ in (2.14). We will not delve any further into the details of these

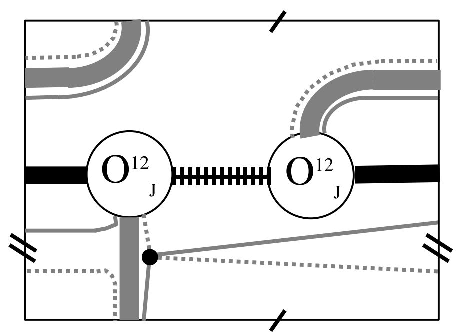


Figure 2.4: A leading g_2 contribution to the anomalous dimension of the operator (2.12) is given by the diagram pictured above. The dotted grey line represents an impurity field which interacts via the quartic vertex with a Z field. The second impurity sits in the central group (indicated by the dashed black line).

gauge theory calculations, and refer the reader to the references [67–73] for details. Suffice it to say that the most important BMN operators concerning this thesis are the two-impurity operators used as examples in this section. The state of the art concerning the full λ' and g_2 expansion of the anomalous dimension may be summarized as

$$\Delta - J = 2 + n^2\lambda' - \frac{1}{4}n^4\lambda'^2 + \frac{1}{8}n^6\lambda'^3 ... + \frac{g_2^2}{4\pi^2}\left(\frac{1}{12} + \frac{35}{32n^2\pi^2}\right)\left(\lambda' - \frac{1}{2}\lambda'^2 n^2\right) + \dots \quad (2.15)$$

where the first set of dots should be understood to mean that the entire square root in (2.6) has in principle been proven in the gauge theory [66].

Reproducing (2.15) from string theory involves reproducing the g_2 terms via the consideration of string interactions on the plane-wave geometry (2.1), in the limit where both λ' and g_2 are small. From our dictionary stemming from (2.5), we have that

$$\lambda' = \frac{g_{YM}^2 N}{J^2} = \frac{1}{(\mu\alpha' p^+)^2}, \qquad g_2 = \frac{J^2}{N} = 4\pi g_s(\mu\alpha' p^+)^2 \quad (2.16)$$

we therefore would like to take the large μ, small g_s limit of the string theory, and calculate the one-loop shift to the energy of a string excited by two oscillators. In order to achieve this, we must have the machinery to calculate general string interactions at our disposal. Such a machinery has been developed in the literature, and the author has made some important contributions to it. In the next section an overview of the state of the art prior to the author's work will be given.

Before we do this it is important to note that the anomalous dimension in (2.15) also receives non-perturbative corrections known as *instanton*[2] corrections. These corrections correspond to similar non-perturbative processes in the dual string theory, arising as contributions of D-branes with point-like worldvolume, the so-called *D-instantons* or D(-1)-branes, see [43]. In a series of important papers [39–42] Green, Sinha, and Kovacs calculated these contributions in both the gauge and string theory finding remarkable agreement between the two.

2.2 Light-cone string field theory on the plane-wave: Introduction

At the start of this chapter (chapter 2), we introduced the basic idea of light-cone string field theory. Indeed, the reader should have in mind figure 2.1. The specific map for the plane-wave superstring was developed originally by Spradlin and Volovich [53, 54] but was revised and elaborated in a long and technically complicated literature [74–88]. This work took its cue directly from the work of the flat-space light-cone string field theory for type IIB superstrings developed in the 1980's by Green, Schwarz, and Brink [45, 46], and elaborated in a series of subsequent papers [47–50]. Rather than trace through the historical development, we will strive to give a self-contained presentation of the state of the art, in order to lay the ground to introduce the author's contributions in subsequent sections. We begin with the free string, and then introduce interactions.

[2]For a review of instantons, see [44].

2.2.1 The free string on the plane-wave background

We begin by adopting an unusual convention for the coordinate length around a closed string. Figure 2.5 depicts the three-string interaction which we are interested in. On the right, this "pair of pants" diagram has been cut and un-folded to reveal the desired parametrization of the three strings; here $\alpha_r \equiv \alpha' p_r^+$, where $r = 1, 2, 3$ labels the string in question. The convention chosen here is that $\sigma \in [-\pi|\alpha_r|, \pi|\alpha_r|]$ for each string. The convention is to further set $\alpha_3 < 0$, while $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 0$ as required by conservation of p^+. This convention obviously ensures a constant p^+-density over the string worldsheet, which turns out to be convenient. The light-cone Green-Schwarz action for the type IIB

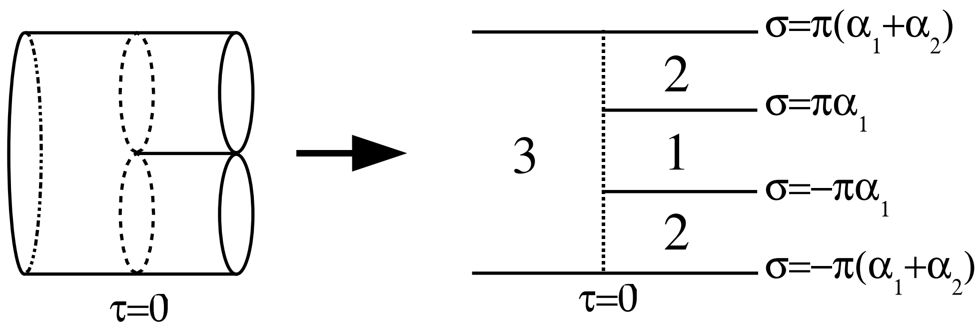


Figure 2.5: The interaction between three closed superstrings is given by the "pair of pants" diagram on the left. Cutting this diagram and unfolding it (as shown on the right) reveals a convenient parametrization of the worldsheet spatial coordinate σ. The lines at $\sigma = \pm\pi(\alpha_1 + \alpha_2)$ are identified, as are those at $\sigma = \pm\pi\alpha_1$. See text for further explanation.

strings in the pp-wave background is then given by [59, 60]

$$S = \frac{e(\alpha)}{4\pi\alpha'} \int d\tau \int_0^{2\pi|\alpha|} d\sigma \left(\partial_\tau X^I \partial_\tau X^I - \partial_\sigma X^I \partial_\sigma X^I - \mu^2 X^I X^I\right) + \\ + \frac{1}{8\pi} \int d\tau \int_0^{2\pi|\alpha|} d\sigma \left(i\bar{\vartheta}\partial_\tau \vartheta + i\vartheta\partial_\tau\bar{\vartheta} - \vartheta\partial_\sigma\vartheta + \bar{\vartheta}\partial_\sigma\bar{\vartheta} - 2\mu\bar{\vartheta}\Pi\vartheta\right) \tag{2.17}$$

where $I = 1, ..., 8$, $e(\alpha) = \text{sign}(\alpha)$, $\alpha = \alpha' p^+$, θ is an 8-component, complex, positive chirality spinor of SO(8), and $\Pi = \gamma^1\gamma^2\gamma^3\gamma^4$ is a symmetric, traceless projection operator, $\Pi^2 = 1$. Here γ^I are the $SO(8)$ Weyl matrices.[3] Notice that the two real spinors analogous to θ^1 and θ^2 in the flat-space case (1.43) have been combined into a complex spinor in which $\vartheta = (\theta^1 + i\theta^2)$ while $\bar{\vartheta} = i(\theta^1 - i\theta^2)$. The equations of motion resulting from the action (2.17) are as follows

[3] The $SO(8)$ gamma-matrices are $\Gamma^I = \left(\begin{pmatrix} 0 & \gamma^I \\ \bar{\gamma}^I & 0 \end{pmatrix}\right)$.

$$\begin{gathered}\left(\partial_\tau^2-\partial_\sigma^2\right)X^I+\mu^2X^I=0\\ i\partial_\tau\bar\vartheta-\partial_\sigma\vartheta+\mu\Pi\bar\vartheta=0,\qquad i\partial_\tau\vartheta+\partial_\sigma\bar\vartheta-\mu\Pi\vartheta=0.\end{gathered}\tag{2.18}$$

The equations (2.18) show that the eight transverse directions form a parabolic "trough" lending a mass $\sim\mu^2$ to the fields X^I,ϑ, see figure 2.6. In this sense massless particles (and strings) race down the light-cone direction x^- (at the bottom of the trough) at the speed of light. In order for a string to (substantially) visit the transverse directions, it requires an excitation on the order of at least μ (in energy)[4]. Thus massive strings extend into the transverse directions x^I. The free, massive, 2-d Klein-Gordon equation (2.18), supplemented by the closed string boundary conditions $X^I(\tau,\sigma+2\pi\alpha)=X^I(\tau,\sigma)$ may be easily solved via the ansätz

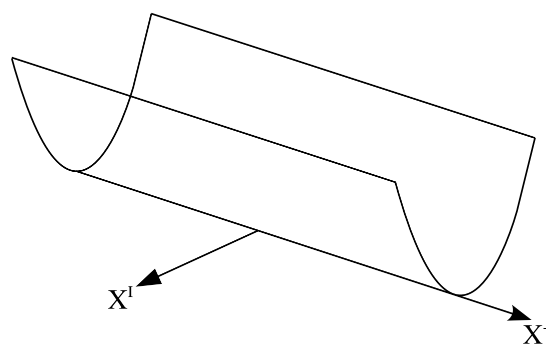


Figure 2.6: The pp-wave geometry (2.1) as viewed by a particle or string. In the transverse directions the string sees a potential $\sim\mu^2{x^I}^2$. In this way only massive (i.e. excited) strings venture appreciably off the light-cone direction x^-, which is a flat direction in the geometry.

$$X^I=\sum_n\left(x_ne^{i\left(\frac{n\sigma}{\alpha}+\epsilon_n\tau\right)}+\widetilde{x}_ne^{i\left(\frac{n\sigma}{\alpha}-\epsilon_n\tau\right)}\right)\tag{2.19}$$

which yields, upon application of (2.18)

$$-\epsilon_n^2+\frac{n^2}{\alpha^2}+\mu^2=0.\tag{2.20}$$

[4]The existence of asymptotically free states in such a potential is by no means guaranteed. Indeed, since only the x^+ and x^- directions are flat, we can only hope to separate wave-packets in this plane. In an important paper by Bak and Sheikh-Jabbari [63], convincing evidence was given for the existence of a 1+1-dimensional S-matrix for massive modes on the plane-wave background. The massless (i.e. supergravity) modes present a curious riddle because they all propagate at the same speed down the trough - they can never catch-up to each other to scatter. This fact appears related to the gauge theory observation that multi-trace operators corresponding to states containing multiple supergravity modes are severly degenerate; for a recent discussion see [64, 65]. The issue of existence of an S-matrix is important for attempts to relate three and four-point functions of BMN operators to their string theory counterparts as in [87] and [88]. The author thanks Mohammad M. Sheikh-Jabbari for pointing this out to him.

To simplify things we define $\omega_n \equiv \sqrt{n^2 + (\mu\alpha)^2}$ so that $\epsilon_n = \omega_n/|\alpha|$. The solution for the fermionic field proceeds in a similar way; at the end of the day we may express the mode expansions for the fields (and their conjugate momenta P^I and λ) at $\tau = 0$ (i.e. where the interaction will be taking place) in the following convenient form

$$X^I(\sigma) = x_0^I + \sqrt{2}\sum_{n=1}^{\infty}\left(x_n^I \cos\frac{n\sigma}{|\alpha|} + x_{-n}^I \sin\frac{n\sigma}{|\alpha|}\right) \tag{2.21}$$

$$P^I(\sigma) = \frac{1}{2\pi|\alpha|}\left[p_0^I + \sqrt{2}\sum_{n=1}^{\infty}\left(p_n^I \cos\frac{n\sigma}{|\alpha|} + p_{-n}^I \sin\frac{n\sigma}{|\alpha|}\right)\right] \tag{2.22}$$

$$\vartheta^a(\sigma) = \vartheta_0^a + \sqrt{2}\sum_{n=1}^{\infty}\left(\vartheta_n^a \cos\frac{n\sigma}{|\alpha|} + \vartheta_{-n}^a \sin\frac{n\sigma}{|\alpha|}\right) \tag{2.23}$$

$$\lambda^a(\sigma) = \frac{1}{2\pi|\alpha|}\left[\lambda_0^a + \sqrt{2}\sum_{n=1}^{\infty}\left(\lambda_n^a \cos\frac{n\sigma}{|\alpha|} + \lambda_{-n}^a \sin\frac{n\sigma}{|\alpha|}\right)\right] \tag{2.24}$$

where $2\lambda_n^a = |\alpha|\bar{\vartheta}_n^a$ and a is an $SO(8)$ spinor index. Quantization then proceeds in a straightforward manner; the non-vanishing (anti-)commutators of the Fourier modes are

$$\left[x_m^I, p_n^J\right] = i\delta^{IJ}\delta_{mn} \quad , \quad \left\{\vartheta_m^a, \lambda_n^b\right\} = \delta^{ab}\delta_{mn} \tag{2.25}$$

and ensure that

$$\left[X^I(\sigma), P^J(\sigma')\right] = i\delta^{IJ}\delta(\sigma - \sigma') \quad , \quad \left\{\vartheta^a(\sigma), \lambda^b(\sigma')\right\} = \delta^{ab}\delta(\sigma - \sigma'). \tag{2.26}$$

The modes can then be written in terms of oscillators

$$x_n^I = i\sqrt{\frac{\alpha'}{2\omega_n}}\left(a_n^I - a_n^{I\dagger}\right) \quad , \quad p_n^I = \sqrt{\frac{\omega_n}{2\alpha'}}\left(a_n^I + a_n^{I\dagger}\right) \quad , \quad \left[a_m^I, a_n^{J\dagger}\right] = \delta^{IJ}\delta_{mn} \tag{2.27}$$

$$\vartheta_n^a = \frac{c_n}{\sqrt{|\alpha|}}\left[(1 + \rho_n\Pi)\, b_n^a + e(n\alpha)\,(1 - \rho_n\Pi)\, b_{-n}^{a\dagger}\right] \tag{2.28}$$

$$\lambda_n^a = \frac{\sqrt{|\alpha|}c_n}{2}\left[(1 + \rho_n\Pi)\, b_n^{a\dagger} + e(n\alpha)\,(1 - \rho_n\Pi)\, b_{-n}^{a}\right] \tag{2.29}$$

$$\left\{b_m^a, b_n^{b\dagger}\right\} = \delta^{ab}\delta_{mn} \tag{2.30}$$

where

$$\rho_n = \frac{\omega_n - |n|}{\mu\alpha}, \qquad c_n = \frac{1}{\sqrt{1 + \rho_n^2}}. \tag{2.31}$$

This rather bizarre transformation of variables for the fermionic oscillators was introduced in [53] in order that the Hamiltonian appear in the canonical form (2.33). The free string Hamiltonian for the r-th string is

$$
\begin{aligned}
H_2^{(r)} = & \frac{e(\alpha)}{2}\int_0^{2\pi|\alpha_r|} d\sigma \left[2\pi\alpha' P^{(r)2} + \frac{1}{2\pi\alpha'}(\partial_\sigma X^{(r)})^2 + \frac{1}{2\pi\alpha'}\mu^2 X^{(r)2})\right] \\
& + \frac{1}{2}\int_0^{2\pi|\alpha_r|} d\sigma \left[-2\pi\alpha'\lambda^{(r)}\partial_\sigma\lambda^{(r)} + \frac{1}{2\pi\alpha'}\theta^{(r)}\partial_\sigma\theta^{(r)} + 2\mu\lambda^{(r)}\Pi\theta^{(r)}\right]
\end{aligned} \tag{2.32}
$$

and in this Fock space basis reduces to

$$
H_2^{(r)} = \sum_{n=-\infty}^{\infty} \frac{\omega_n^{(r)}}{\alpha_r}\left(a_n^{(r)I\dagger} a_n^{(r)I} + b_n^{(r)a\dagger} b_n^{(r)a}\right). \tag{2.33}
$$

The states of the (single) string are then built upon its vacuum $|0;\alpha\rangle$ (recall that $\alpha = \alpha' p^+$) which is annihilated by the lowering operators

$$
a_n|0;\alpha\rangle = b_n|0;\alpha\rangle = 0, \qquad \forall n. \tag{2.34}
$$

Physical string states $|\Psi\rangle$ are built by acting the $a_n^\dagger$ and $b_n^\dagger$ upon this vacuum, subject to the level-matching condition (1.29)

$$
\sum_n n\left(\delta_{IJ} a_{-n}^{I\dagger} a_n^J + \delta_{ab} b_{-n}^{a\dagger} b_n^b\right)|\Psi\rangle = 0. \tag{2.35}
$$

Because the basis (2.28) breaks the $SO(8)$ symmetry of the plane-wave to $SO(4) \times SO(4) \times \mathbb{Z}_2$, it will be easier to introduce a new basis for the γ-matrices in which [84]

$$
\Pi = \begin{pmatrix} \delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} & 0 \\ 0 & -\delta_{\dot\beta_1}^{\dot\alpha_1}\delta_{\dot\beta_2}^{\dot\alpha_2} \end{pmatrix} \tag{2.36}
$$

where we label representations of $SO(4)_1 \times SO(4)_2$ through $(SU(2)\times SU(2))_1 \times (SU(2) \times SU(2))_2$ spinor indices. With this decomposition of the R-charge index, the fermionic fields ϑ^a and λ^a, are expressed in terms of creation operators $b^\dagger_{\alpha_1\alpha_2}$ and $b^\dagger_{\dot\alpha_1\dot\alpha_2}$ which transform in the $(1/2,0,1/2,0)$ and $(0,1/2,0,1/2)$ representations of $(SU(2)\times SU(2))_1 \times (SU(2)\times SU(2))_2$, respectively; α_k,$\dot\alpha_k$ being two-component Weyl indices of $SO(4)_k$, see appendix A for details. The Hamiltonian (2.33) in this basis is then

$$
H_2^{(r)} = \sum_{n=-\infty}^{\infty} \frac{\omega_n^{(r)}}{\alpha_r}\left(a_n^{(r)I\dagger} a_n^{(r)I} + b_{n(r)}^{\alpha_1\alpha_2\dagger} b_{n(r)\,\alpha_1\alpha_2} + b_{n(r)}^{\dot\alpha_1\dot\alpha_2\dagger} b_{n(r)\,\dot\alpha_1\dot\alpha_2}\right) \tag{2.37}
$$

while the commutation relations are given by

$$
\left\{b_{n(r)\,\alpha_1\alpha_2}, b_{m(s)}^{\beta_1\beta_2\dagger}\right\} = \delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2}\delta_{mn}\delta_{rs}, \qquad \left\{b_{n(r)\,\dot\alpha_1\dot\alpha_2}, b_{m(s)}^{\dot\beta_1\dot\beta_2\dagger}\right\} = \delta_{\dot\alpha_1}^{\dot\beta_1}\delta_{\dot\alpha_2}^{\dot\beta_2}\delta_{mn}\delta_{rs}. \tag{2.38}
$$

2.2.2 Local and non-local isometries

The super-Poicaré group of isometries respected by the plane-wave involve the following generators

$$\begin{gathered} P^+, P^I, J^{+I}, J^{ij}, J^{i'j'}, Q^+, \bar{Q}^+ \\ H, Q^-, \bar{Q}^- \end{gathered} \tag{2.39}$$

where $i, j = 1, \ldots, 4$ and $i', j' = 5, \ldots, 8$. These are the same objects we saw in section 1.4.4 where J^{IJ} here corresponds to M^{IJ} used there. The commutation relations which are different from those of flat space are given by

$$\begin{gathered} [H, P^I] = i\mu^2 J^{+I}, \qquad [P^I, Q^-] = \mu \Pi \gamma^I Q^+, \qquad [H, Q^+] = \mu \Pi Q^+, \\ \left\{Q^-, \bar{Q}^-\right\} = 2H + i\mu\gamma_{ij}\Pi J^{ij} + i\mu\gamma_{i',j'}\Pi J^{i'j'}. \end{gathered} \tag{2.40}$$

The generators (2.39) fall into two fundamentally different categories; those on the top line and those on the bottom line. The generators on the top line are *local*, or *kinematical* generators. This means that they act at a point on the string world sheet. Consequently, they are not capable of splitting or joining strings as this is a clearly non-local operation (see figure 2.5). Conversely, the Hamiltonian H and the supercharges Q^- and $\bar{Q}^-$ are non-local. The Hamiltonian involves naturally a derivative of the light-cone time x^+, while the supercharges involve σ derivatives. These non-local or *dynamical* generators do induce string interactions and therefore receive corrections from those interactions. The kinematical generators remain unchanged by string interactions, and this allows us to largely forget about them.

The free string Hamiltonian was given in (2.33). The free string supercharges are given formally by

$$Q^+_{(r)} = \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi|\alpha_r|} d\sigma_r \, \sqrt{2}\lambda_r \,, \tag{2.41}$$

$$Q^-_{(r)} = \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi|\alpha_r|} d\sigma_r \, \left[2\pi\alpha' e(\alpha_r) p_r \gamma\lambda_r - i x'_r \gamma\bar{\lambda}_r - i\mu x_r \gamma\Pi\lambda_r\right] \tag{2.42}$$

where $\bar{Q}^\pm_{(r)} = e(\alpha_r)\left[Q^\pm_{(r)}\right]^\dagger$. Plugging the mode expansions of the fields into the expression for Q^-, we obtain

$$\begin{aligned} Q^-_{(r)} &= \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\gamma\Big(\sqrt{\mu}\left[a_{0(r)}(1+e(\alpha_r)\Pi) + a^\dagger_{0(r)}(1-e(\alpha_r)\Pi)\right]\lambda_{0(r)} \\ &+ \sum_{n\neq 0}\sqrt{|n|}\left[a_{n(r)}P^{-1}_{n(r)}b^\dagger_{n(r)} + e(\alpha_r)e(n)a^\dagger_{n(r)}P_{n(r)}b_{-n(r)}\right]\Big)\,, \end{aligned} \tag{2.43}$$

where

$$P_{n(r)} \equiv \frac{1-\rho_{n(r)}\Pi}{\sqrt{1-\rho_{n(r)}^2}} = \frac{1+\Pi}{2}U_{|n|(r)}^{1/2} + \frac{1-\Pi}{2}U_{|n|(r)}^{-1/2}, \qquad U_{n(r)} \equiv \frac{\omega_{n(r)}-\mu\alpha_r}{n}\,. \tag{2.44}$$

As advertised, Q^- and H will be corrected beyond the free string result by string interactions. In the next section we will trace the construction of these corrections.

2.2.3 The string field and determination of the interaction vertices

Light-cone string field theory, as its name implies, deals with a *string field* - a field which strings are elementary excitations of. Consider a bosonic string, it has an infinite number of modes, here labelled by k, and each is a harmonic oscillator with creation operator $a_k^\dagger$. In order to specify a completely general string state, one must specify the occupation numbers $\{n_k\}$, giving the level of excitation (i.e. the number of $a_k^\dagger$'s) for each mode. There is also the further multiplicity of the spacetime dimensions, which we will here label by i, and so we should in fact say that a string will be entirely specified by the set $\{n_{k\,i}\}$. Instead of relying on this *number basis* to define the string field, we would rather like to describe it as a momentum distribution. Indeed, the mode expansion of the conjugate momentum $P(\sigma)$ involves the set $\{p_{k\,i}\}$ of Fourier coefficients, c.f. (2.22). We can then define a string field $\Phi[P(\sigma)]$, which acting on the vacuum creates a string (at $\tau = 0$) defined by it's conjugate momentum $P(\sigma)$. The expression may be given as

$$\Phi[P(\sigma)] = \sum_{\{n_{k\,i}\}} \varphi_{\{n_{k\,i}\}} \prod_{k=1}^{\infty}\prod_i \psi_{n_{k\,i}}(p_{k\,i}) \tag{2.45}$$

where $\varphi_{\{n_{k\,i}\}}$ creates the number basis state given by $\{n_{k\,i}\}$

$$\varphi_{\{n_{k\,i}\}} = \prod_{k=1}^{\infty}\prod_i \left(a_k^{i\dagger}\right)^{n_{k\,i}} \tag{2.46}$$

while $\psi_{n_{k\,i}}(p_{k\,i})$ is the momentum space wavefunction of the k_i^{th} simple harmonic oscillator in the $n_{k\,i}^{\text{th}}$ excited state

$$\psi_n(p) = \langle n|p\rangle, \qquad |p\rangle \sim \exp\left(-\frac{1}{4}p^2 + p\,a^\dagger - \frac{1}{2}a^\dagger a^\dagger\right)|0\rangle. \tag{2.47}$$

Finally the sum in (2.45) is over all possible combinations of occupation numbers - i.e. all physical states of the string. The string field operator $\Phi[P(\sigma)]$ has the property that

$$\widehat{P}(\sigma)\Phi[P(\sigma)]|0\rangle = P(\sigma)\Phi[P(\sigma)]|0\rangle \tag{2.48}$$

that is, $\Phi[P(\sigma)]|0\rangle$ is an eigenstate of the total momentum operator $\widehat{P}(\sigma)$. This construction may be easily generalized to include fermionic modes created by $b_k^{a\dagger}$. Rather than explicitly tracing the fermionic construction here, we will continue with the bosonic modes and construct the first correction to the Hamiltonian.

As was mentioned in section 2.2.2, the Hamiltonian is a dynamical generator which we expect to be corrected by string interactions. Let κ be the coupling constant controlling string interactions, we expect that

$$H = H_2 + \kappa H_3 + \kappa^2 H_4 + \dots \tag{2.49}$$

where H_n is an operator which maps one string to $n-1$ strings; i.e. an operator involving a product of n string fields Φ. The quadratic Hamiltonian H_2 is simply the free string one given in (2.33). In constructing H_3, we will be guided by symmetry. The plane-wave background is not translationally invariant in the eight transverse directions, as is plain from figure 2.6, or from the first commutation relation in (2.40). However, somewhat miraculously, the commutator of H and P^I is proportional to a *kinematical* generator. This means that although $[H_2, P^I] \neq 0$, we have that $[H_{n>2}, P^I] = 0$ as it is in the flat space case. From the point of view of the interaction Hamiltonian, the transverse momentum is conserved. A natural ansätz for the cubic Hamiltonian is then

$$\begin{aligned} H_3 &= \int d\mathcal{M}\, h_3\, \Phi[P_1(\sigma)]\, \Phi[P_2(\sigma)]\, \Phi[P_3(\sigma)], \\ d\mathcal{M} &= \left(\prod_{r=1}^{3} d\alpha_r\, DP_r(\sigma)\right) \delta\left(\sum \alpha_r\right) \delta\left(\sum P_r(\sigma)\right) \end{aligned} \tag{2.50}$$

where h_3 is called the "prefactor" and is an as yet undetermined function. The object is now to reduce (2.50) into an expression involving only the oscillators of the strings. The first step in this direction is accomplished by representing the delta function enforcing conservation of transverse momentum P^I in the Fourier basis of the third string

$$\Delta[f(\sigma)] = \prod_m \delta\left(\int_0^{2\pi|\alpha_3|} e^{im\sigma/|\alpha_3|} f(\sigma)\, d\sigma\right) \tag{2.51}$$

so that, for example

$$\begin{aligned} \Delta[P_2(\sigma)] = \prod_{m=-\infty}^{\infty} \delta\Bigg(\int_0^{2\pi|\alpha_3|} d\sigma\, e^{im\sigma/|\alpha_3|} \frac{1}{2\pi|\alpha_2|} \\ \times \left[p_0^{(2)I} + \sqrt{2} \sum_{n=1}^{\infty} \left(p_n^{(2)I} \cos\frac{n\sigma}{|\alpha_2|} + p_{-n}^{(2)I} \sin\frac{n\sigma}{|\alpha_2|} \right) \right] \Theta\left(|\sigma| - \pi\alpha_1\right) \Bigg) \end{aligned} \tag{2.52}$$

where the Heaviside function $\Theta\left(|\sigma| - \pi\alpha_1\right)$ enforces the limits of the second string's world-sheet (see figure 2.5). Performing the integral over σ, and similarly for strings 1 and 3, we arrive at

$$\delta\left(\sum P_r(\sigma)\right) = \Delta\left[\sum_{r=1}^{3} P_r\right] = \prod_{I=1}^{8} \prod_{m=-\infty}^{\infty} \delta\left(p_m^{(3)I} + \sum_{n=-\infty}^{\infty} \left(X_{mn}^{(1)}\, p_n^{(1)I} + X_{mn}^{(2)}\, p_n^{(2)I}\right)\right) \tag{2.53}$$

where the matrices $X^{(r)}_{mn}$ perform the transformation between the Fourier basis of the third string and those of strings 1 and 2, so that $X^{(3)}_{mn} = \mathbf{1}$. Next we turn to the product of three string fields in (2.50). In fact the definition of the string field (2.45) is slightly redundant. Consider the simple one-dimensional simple harmonic oscillator. The following sum has a simple form

$$\sum_n |n\rangle\psi_n(p) = \sum_n |n\rangle\langle n|p\rangle = |p\rangle \tag{2.54}$$

in fact (2.45) is nothing but a generalization of this same form. This allows us to write the product of string fields as

$$\begin{aligned}
&\prod_r \Phi[P_r] = \mathcal{N}\exp\left[\sum_{n,r,I}\left(-\frac{1}{\omega_{n(r)}}\left(p^{(r)I}_n\right)^2 + \frac{2}{\sqrt{\omega_{n(r)}}}p^{(r)I}_n a^{(r)I\dagger}_n - \frac{1}{2}a^{(r)I\dagger}_n a^{(r)I\dagger}_n\right)\right], \\
&\text{with} \qquad \mathcal{N} = \prod_{n,r,I}\left(\frac{2}{\pi\omega_{n(r)}}\right)^{1/4}
\end{aligned} \tag{2.55}$$

where we have used the correctly normalized wavefunctions appropriate to the string on the plane-wave, i.e. correct version of (2.47). Postponing a discussion of the prefactor h_3 until later, we can now proceed with the Gaussian integration resulting from plugging (2.55) and (2.53) into (2.50). The integration proceeds over the transverse momentum modes, taking (for now) $h_3 = 1$, and leaving the integration over the light-cone momentum α_r until later. The result may be summarized as follows

$$\int\left(\prod_{n,r,I} dp^{(r)I}_n\right)\prod_r \Phi[P_r]\,\Delta\left[\sum_{r=1}^{3} P_r\right] = \mathcal{C}\exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} a^{(r)I\dagger}_m \bar{N}^{rs}_{mn} a^{(s)I\dagger}_n\right) \tag{2.56}$$

where $\mathcal{C}$ is an overall constant which won't be important for us, while N^{rs}_{mn} are known as the "Neumann matrices" and may be expressed in terms of the $X^{(r)}_{mn}$ as [53]

$$\bar{N}^{rs}_{mn} = \delta^{rs}\delta_{mn} - 2\sqrt{\omega_{m(r)}\omega_{n(s)}}\left(X^{(r)^T}\Gamma^{-1}X^{(s)}\right)_{mn}, \quad \Gamma_{mn} = \sum_r\sum_{p=-\infty}^{\infty}\omega_{p(r)}X^{(r)}_{mp}X^{(r)}_{np}. \tag{2.57}$$

At this point it is useful to step back and summarize what we have found. The expression (2.56) is the oscillator manifestation of transverse momentum conservation. Interpreted from a spatial point of view, the oscillator map (2.56) ensures that the three strings touch at $\tau = 0$, the moment of the interaction. There are however, other symmetries which the interaction Hamiltonian should satisfy. In the same way that P^I commuted with the interacting piece of the Hamiltonian, a look at (2.40) reveals that $[H_{n>2}, Q^+] = 0$ since the full commutator is proportional to a kinematical generator (Q^+ itself). Of course the string field (2.45) also needs to be amended to reflect the fermionic modes of the superstring. Including the fermionic modes and enforcing the $[H_{n>2}, Q^+] = 0$ symmetry is quite literally the supersymmetric reflection of the bosonic construction which was detailed in the previous paragraph. The details of the construction may be gleaned from [53, 79, 84], the result is that the fermionic equivalent of (2.56) is

$$\exp\left(\sum_{r,s=1}^{3}\sum_{m,n\geq 0}\left(b^{\alpha_1\alpha_2\,\dagger}_{-m(r)}b_{n(s)\,\alpha_1\alpha_2}+b^{\dot\alpha_1\dot\alpha_2\,\dagger}_{-m(r)}b_{n(s)\,\dot\alpha_1\dot\alpha_2}\right)\bar{Q}^{rs}_{mn}\right) \tag{2.58}$$

where Q^{rs}_{mn} is a fermionic Neumann matrix which will be given explicitly later. The progenitor of an interaction vertex $|V\rangle$ may then be built as follows

$$\begin{aligned}|V\rangle =&\delta\left(\sum\alpha_r\right)\exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}a^{(r)I\dagger}_{m}\bar{N}^{rs}_{mn}a^{(s)I\dagger}_{n}\right)\\ &\times\exp\left(\sum_{r,s=1}^{3}\sum_{m,n\geq 0}\left(b^{\alpha_1\alpha_2\,\dagger}_{-m(r)}b_{n(s)\,\alpha_1\alpha_2}+b^{\dot\alpha_1\dot\alpha_2\,\dagger}_{-m(r)}b_{n(s)\,\dot\alpha_1\dot\alpha_2}\right)\bar{Q}^{rs}_{mn}\right)|0;\alpha_1\rangle\otimes|0;\alpha_2\rangle\otimes|0;\alpha_3\rangle.\end{aligned} \tag{2.59}$$

There is a subtlety here concerning the fact that α_3 is negative. It concerns the definition of the adjoint for string #3. Already it should have seemed suspicious that the free Hamiltonian (2.33) is not strictly positive, since $\alpha_3 < 0$. In fact the adjoint on the full Hilbert space $\mathcal{H} = \oplus_m\mathcal{H}_m$, where $\mathcal{H}_m$ is the m-string Hilbert space, is not the same as the single string Hilbert space adjoint [56][53]. Objects such as V in (2.59) may be viewed as operators from $\mathcal{H}_1 \to \mathcal{H}_2$, or as states in $\mathcal{H}_3$

$$\langle 3|V|2\rangle|1\rangle \equiv \langle 1|\langle 2|\langle 3'|V\rangle. \tag{2.60}$$

The prime denotes the fact that the adjoint on string #3 is modified by a sign, so that, for example, if $|\phi^{(3)}\rangle$ is some state built on $|0;\alpha_3\rangle$ so that $|\lambda^{(3)}\rangle = A^{(3)}|\phi^{(3)}\rangle$ where $A^{(3)}$ is some one-string operator (i.e. from $\mathcal{H}_1 \to \mathcal{H}_1$), then

$$\langle\lambda^{(3)}||2\rangle|1\rangle \equiv \langle\phi^{(3)}|(-A^{(3)\dagger})|2\rangle|1\rangle \tag{2.61}$$

whereas this sign is absent for strings #1 and #2, as α_1 and α_2 are taken positive. This ensures the positivity of the free string energy

$$\langle H_2\rangle = \langle 3|H_2|2\rangle|1\rangle = \sum_{r=1}^{3}e(\alpha_r)H_2^{(r)} > 0. \tag{2.62}$$

Below we will construct the cubic Hamiltonian and supercharges as states in $\mathcal{H}_3$. The subtlety (2.61) will only arise when considering operators in string #3's Hilbert space $\mathcal{H}_1^{(3)}$.

The vertex (2.59) respects the super-locality symmetry, but there is one last symmetry which we have yet to enforce. That is the commutation relations between the Q's given on the second line of (2.40). We see from (2.40) that Q^- and $\bar{Q}^-$, like H, commute with P^I and Q^+ to give kinematical generators. Therefore we can build Q_3^- and $\bar{Q}_3^-$ using the progenitor vertex $|V\rangle$ as well. Recall that we included a prefactor h_3 in our definition (2.50), which we then set to 1 and forgot about. The idea is to now restore this prefactor (and similar q_3^- and $\bar{q}_3^-$ for the supercharges) via an operator acting on $|V\rangle$

$$|H_3\rangle = h_3|V\rangle, \qquad |Q_3^-\rangle = q_3^-|V\rangle, \qquad |\bar{Q}_3^-\rangle = \bar{q}_3^-|V\rangle \tag{2.63}$$

and then to determine the specific form of the prefactors by ensuring the closure of the supersymmetry algebra (2.40). This process is simplified in the (2.38) basis for the fermions. There, as shown in appendix A, linear combinations of Q_3^- and $\bar{Q}_3^-$ may be taken so that

$$\left\{Q_{\alpha_1\dot{\alpha}_2}, Q_{\beta_1\dot{\beta}_2}\right\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H \tag{2.64}$$

i.e. we can factor away the dependence on J^{ij} and $J^{i'j'}$ in (2.40). At first order in κ, we will have schematically $\{Q_2, Q_3\} \sim H_3$. Written using the state language, as in (2.63), we have

$$\sum_{r=1}^{3} Q_{(r)\,\alpha_1\dot{\alpha}_2}|Q_{3\,\beta_1\dot{\beta}_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\beta_1\dot{\beta}_2}|Q_{3\,\alpha_1\dot{\alpha}_2}\rangle = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}|H_3\rangle\,, \tag{2.65}$$

$$\sum_{r=1}^{3} Q_{(r)\,\dot{\alpha}_1\alpha_2}|Q_{3\,\dot{\beta}_1\beta_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\dot{\beta}_1\beta_2}|Q_{3\,\dot{\alpha}_1\alpha_2}\rangle = -2\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\alpha_2\beta_2}|H_3\rangle\,, \tag{2.66}$$

$$\sum_{r=1}^{3} Q_{(r)\,\alpha_1\dot{\alpha}_2}|Q_{3\,\dot{\beta}_1\beta_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\dot{\beta}_1\beta_2}|Q_{3\,\alpha_1\dot{\alpha}_2}\rangle = 0 \tag{2.67}$$

where $Q_{(r)\,\beta_1\dot{\beta}_2}$ and $Q_{(r)\,\dot{\beta}_1\beta_2}$ are the quadratic, free string supercharges Q_2. Finding a solution for the prefactors which obeys these relations is a non-trivial (and non-unique) undertaking. We will not step the reader through this process, and instead refer to the literature [84], where the following result is obtained

$$\begin{aligned}
|H_3\rangle &= g_2\, f(\mu\alpha_3\,, \frac{\alpha_1}{\alpha_3})\frac{\alpha'}{8\,\alpha_3^3}\Big[\big(K_i\widetilde{K}_j - \frac{\mu\kappa}{\alpha'}\delta_{ij}\big)v^{ij} - \big(K_{i'}\widetilde{K}_{j'} - \frac{\mu\kappa}{\alpha'}\delta_{i'j'}\big)v^{i'j'} \\
&\quad - K^{\dot{\alpha}_1\alpha_1}\widetilde{K}^{\dot{\alpha}_2\alpha_2}s_{\alpha_1\alpha_2}(Y)s^*_{\dot{\alpha}_1\dot{\alpha}_2}(Z) - \widetilde{K}^{\dot{\alpha}_1\alpha_1}K^{\dot{\alpha}_2\alpha_2}s^*_{\alpha_1\alpha_2}(Y)s_{\dot{\alpha}_1\dot{\alpha}_2}(Z)\Big]|V\rangle\,, \\
|Q_{3\,\beta_1\dot{\beta}_2}\rangle &= g_2\,\eta\, f(\mu\alpha_3\,, \frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s_{\dot{\gamma}_1\dot{\beta}_2}(Z)t_{\beta_1\gamma_1}(Y)\widetilde{K}^{\dot{\gamma}_1\gamma_1} \\
&\qquad + is_{\beta_1\gamma_2}(Y)t^*_{\dot{\beta}_2\dot{\gamma}_2}(Z)\widetilde{K}^{\dot{\gamma}_2\gamma_2}\Big)|V\rangle\,, \\
|Q_{3\,\dot{\beta}_1\beta_2}\rangle &= g_2\,\bar{\eta}\, f(\mu\alpha_3\,, \frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s^*_{\gamma_1\beta_2}(Y)t^*_{\dot{\beta}_1\dot{\gamma}_1}(Z)\widetilde{K}^{\dot{\gamma}_1\gamma_1} \\
&\qquad + is^*_{\dot{\beta}_1\dot{\gamma}_2}(Z)t_{\beta_2\gamma_2}(Y)\widetilde{K}^{\dot{\gamma}_2\gamma_2}\Big)|V\rangle\,.
\end{aligned} \tag{2.68}$$

where $\kappa \equiv \alpha_1\alpha_2\alpha_3$, K^I, $\widetilde{K}^I$ are expressions linear in bosonic oscillators and are defined in (2.82), while Y and Z are their fermionic counter-parts and are given in (2.83). Also note that κ, the coupling constant, has been replaced by g_2 (2.16). The string coupling must be this value according to the AdS/CFT correspondence; it cannot be fixed by first principles, hence it is a matter of choice to set $\kappa = g_2$. Further

$$K^{\dot\gamma_1\gamma_1} \equiv K^i \sigma^{i\,\dot\gamma_1\gamma_1}\,, \quad K^{\dot\gamma_2\gamma_2} \equiv K^{i'} \sigma^{i'\,\dot\gamma_2\gamma_2}\,, \quad \widetilde{K}^{\dot\gamma_1\gamma_1} \equiv \widetilde{K}^i \sigma^{i\,\dot\gamma_1\gamma_1}\,, \quad \widetilde{K}^{\dot\gamma_2\gamma_2} \equiv \widetilde{K}^{i'} \sigma^{i'\,\dot\gamma_2\gamma_2} \tag{2.69}$$

where the σ-matrices are defined in appendix A. We also have

$$\begin{aligned}
v^{ij} &= \delta^{ij}\Big[1 + \frac{1}{12}\big(Y^4 + Z^4\big) + \frac{1}{144}Y^4 Z^4\Big] \\
&\quad - \frac{i}{2}\Big[Y^{2ij}\big(1 + \frac{1}{12}Z^4\big) - Z^{2ij}\big(1 + \frac{1}{12}Y^4\big)\Big] + \frac{1}{4}\big[Y^2 Z^2\big]^{ij}\,, \\
v^{i'j'} &= \delta^{i'j'}\Big[1 - \frac{1}{12}\big(Y^4 + Z^4\big) + \frac{1}{144}Y^4 Z^4\Big] \\
&\quad - \frac{i}{2}\Big[Y^{2i'j'}\big(1 - \frac{1}{12}Z^4\big) - Z^{2i'j'}\big(1 - \frac{1}{12}Y^4\big)\Big] + \frac{1}{4}\big[Y^2 Z^2\big]^{i'j'}\,.
\end{aligned}$$

Here we defined

$$Y^{2ij} \equiv \sigma^{ij}_{\alpha_1\beta_1} Y^{2\alpha_1\beta_1}\,, \quad Z^{2ij} \equiv \sigma^{ij}_{\dot\alpha_1\dot\beta_1} Z^{2\dot\alpha_1\dot\beta_1}\,, \quad \big(Y^2 Z^2\big)^{ij} \equiv Y^{2k(i} Z^{2j)k} \tag{2.70}$$

and analogously for the primed indices. We have also introduced the following quantities quadratic and cubic in Y and symmetric in spinor indices

$$Y^2_{\alpha_1\beta_1} \equiv Y_{\alpha_1\alpha_2} Y^{\alpha_2}_{\beta_1}\,, \qquad Y^2_{\alpha_2\beta_2} \equiv Y_{\alpha_1\alpha_2} Y^{\alpha_1}_{\beta_2} \tag{2.71}$$

$$Y^3_{\alpha_1\beta_2} \equiv Y^2_{\alpha_1\beta_1} Y^{\beta_1}_{\beta_2} = -Y^2_{\beta_2\alpha_2} Y^{\alpha_2}_{\alpha_1}\,, \tag{2.72}$$

and quartic in Y and antisymmetric in spinor indices

$$Y^4_{\alpha_1\beta_1} \equiv Y^2_{\alpha_1\gamma_1} Y^{2\gamma_1}{}_{\beta_1} = -\frac{1}{2}\epsilon_{\alpha_1\beta_1} Y^4\,, \qquad Y^4_{\alpha_2\beta_2} \equiv Y^2_{\alpha_2\gamma_2} Y^{2\gamma_2}{}_{\beta_2} = \frac{1}{2}\epsilon_{\alpha_2\beta_2} Y^4 \tag{2.73}$$

where

$$Y^4 \equiv Y^2_{\alpha_1\beta_1} Y^{2\alpha_1\beta_1} = -Y^2_{\alpha_2\beta_2} Y^{2\alpha_2\beta_2}\,. \tag{2.74}$$

The spinorial quantities s and t are defined as

$$s(Y) \equiv Y + \frac{i}{3}Y^3\,, \qquad t(Y) \equiv \epsilon + iY^2 - \frac{1}{6}Y^4\,. \tag{2.75}$$

Analogous definitions can be given for Z. The normalization of the dynamical generators is not fixed by the superalgebra at $\mathcal{O}(g_2)$ and can be an arbitrary (dimensionless) function $f(\mu\alpha_3\,, \frac{\alpha_1}{\alpha_3})$ of the light-cone momenta and μ due to the fact that P^+ is a central element of the algebra.

The definitions of the quantities Y, Z, K, and $\widetilde{K}$, along with the bosonic and fermionic Neumann matrices are most easily expressed in the so-called "BMN basis" for the oscillators

$$\begin{aligned}
&\sqrt{2}a^i_n \equiv \alpha^i_n + \alpha^i_{-n}\,, \quad i\sqrt{2}a^i_{-n} \equiv \alpha^i_n - \alpha^i_{-n}\\
&\sqrt{2}a^{i'}_n \equiv \alpha^{i'}_n + \alpha^{i'}_{-n}\,, \quad i\sqrt{2}a^{i'}_{-n} \equiv \alpha^{i'}_n - \alpha^{i'}_{-n}\,,\\
&\sqrt{2}b^{\alpha_1\alpha_2}_n \equiv \beta^{\alpha_1\alpha_2}_n + \beta^{\alpha_1\alpha_2}_{-n}\,, \quad i\sqrt{2}b^{\alpha_1\alpha_2}_{-n} \equiv \beta^{\alpha_1\alpha_2}_n - \beta^{\alpha_1\alpha_2}_{-n}\,,\\
&i\sqrt{2}b^{\dot\alpha_1\dot\alpha_2}_n \equiv -\beta^{\dot\alpha_1\dot\alpha_2}_n + \beta^{\dot\alpha_1\dot\alpha_2}_{-n}\,, \quad \sqrt{2}b^{\dot\alpha_1\dot\alpha_2}_{-n} \equiv \beta^{\dot\alpha_1\dot\alpha_2}_n + \beta^{\dot\alpha_1\dot\alpha_2}_{-n}
\end{aligned} \tag{2.76}$$

for $n > 0$, and

$$a^i_0 \equiv \alpha^i_0 \quad a^{i'}_0 \equiv \alpha^{i'}_0 \quad b^{\alpha_1\alpha_2}_0 \equiv \beta^{\alpha_1\alpha_2}_0 \quad b^{\dot\alpha_1\dot\alpha_2}_0 \equiv \beta^{\dot\alpha_1\dot\alpha_2}_0 \tag{2.77}$$

for $n = 0$. The commutation relations for the oscillators are then

$$[\alpha^i_m, \alpha^{\dagger j}_n] = \delta_{mn}\delta^{ij}\,, \quad \{\big(\beta_m\big)_{\alpha_1\alpha_2}, \big(\beta^\dagger_n\big)^{\beta_1\beta_2}\} = \delta_{mn}\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}\,. \tag{2.78}$$

In order to perform the string-field theory calculations we are interested in comparing to gauge theory, we require the large-μ limit of all quantities. These were worked out in [80] and are given in appendix B. We find simpler expressions for them, which are summarized in the BMN basis as

$$|V\rangle = |E_\alpha\rangle|E_\beta\rangle\delta(\sum_{r=1}^{3}\alpha_r) \tag{2.79}$$

where $|E_\alpha\rangle$ and $|E_\beta\rangle$ are exponentials of bosonic and fermionic oscillators respectively

$$|E_\alpha\rangle = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\alpha^{\dagger K}_{m\,(s)}\widetilde{N}^{st}_{mn}\alpha^{\dagger K}_{n\,(t)}\right)|\alpha\rangle_{123} \tag{2.80}$$

and

$$|E_\beta\rangle = \exp\left(\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\big(\beta^{\alpha_1\alpha_2\,\dagger}_{m(r)}\beta^\dagger_{n(s)\,\alpha_1\alpha_2} - \beta^{\dot\alpha_1\dot\alpha_2\,\dagger}_{m(r)}\beta^\dagger_{n(s)\,\dot\alpha_1\dot\alpha_2}\big)\widetilde{Q}^{rs}_{mn}\right)|\alpha\rangle_{123} \tag{2.81}$$

where $|\alpha\rangle_{123} = |0;\alpha_1\rangle \otimes |0;\alpha_2\rangle \otimes |0;\alpha_3\rangle$. We further have that

$$K^I = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} K_{n(s)}\alpha^{I\,\dagger}_{n(s)}\,, \quad \widetilde{K}^I = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} K_{n(s)}\alpha^{I\,\dagger}_{-n(s)} \tag{2.82}$$

$$Y^{\alpha_1\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} G_{|n|(s)}\beta^{\dagger\,\alpha_1\alpha_2}_{n(s)}\,, \qquad Z^{\dot\alpha_1\dot\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} G_{|n|(s)}\beta^{\dagger\,\dot\alpha_1\dot\alpha_2}_{n(s)}\,, \tag{2.83}$$

where the large-μ limits of these quantities are repackaged from the expressions found in appendix B and are expressed as [5]

[5] These expressions (2.84-2.88) are also valid for $q, p = 0$, except in the case of $\widetilde{N}^{rs}_{00} = -\widetilde{N}^{rs}_{qp}|_{q,p=0}$, and in the case of $\widehat{Q}^{3r}_{00} = -ir^{-1}\beta_r\widehat{Q}^{3r}_{np}|_{n,p=0}$.

$$\widetilde{N}^{3\,r}_{n\,q} = -\frac{\sin(n\pi r)\sqrt{\beta_r}\left(\Lambda^+_n\Lambda^+_q + \Lambda^-_n\Lambda^-_q\right)}{2\pi\sqrt{\omega_n\omega_q}\,(q-\beta_r n)}, \qquad \widetilde{N}^{r\,s}_{q\,p} = \frac{\sqrt{\beta_r\beta_s}\left(\Lambda^+_q\Lambda^+_p + \Lambda^-_q\Lambda^-_p\right)}{4\pi\sqrt{\omega_q\omega_p}\,(\beta_s\omega_q+\beta_r\omega_p)}, \tag{2.84}$$

$$\widehat{Q}^{3\,r}_{n\,q} = \frac{i\sin(|n|\pi r)\,(\omega_q+\beta_r\omega_n)}{2\pi\sqrt{\omega_n\omega_q}\,(q-\beta_r n)}, \qquad \widehat{Q}^{r\,s}_{q\,p} = \frac{i\,(\beta_s q - \beta_r p)}{4\pi\sqrt{\omega_q\omega_p}\,(\beta_s\omega_q+\beta_r\omega_p)} \tag{2.85}$$

where $\widehat{Q}^{s\,r}_{n\,q} = \widetilde{Q}^{s\,r}_{n\,q} - \widetilde{Q}^{r\,s}_{q\,n}$, $\beta_r \equiv -\alpha_r/\alpha_3$ for $r = 1,2$, and where we remind the reader that $\alpha_3 < 0$ while $\alpha_1, \alpha_2 > 0$. Also $r \equiv \beta_1$ while $\beta_2 = 1 - r$. The mode number n is associated with string 3, while p and q are used for either string 1 or 2. We also drop the string label on ω_q, K_q, G_q etc. as it is obvious from the quantity given. For example ω_q in $\widetilde{N}^{3r}_{nq}$ should be understood as $\omega^{(r)}_q$. Continuing, we also have

$$K_n = +\alpha_3\sin(n\pi r)\sqrt{\frac{r(1-r)}{\pi\alpha'}}\,\frac{\Lambda^-_n - \Lambda^+_n}{\sqrt{\omega_n}}, \tag{2.86}$$

$$K_q = -\alpha_3\sqrt{\frac{r(1-r)}{\pi\alpha'\beta_r}}\,\frac{\Lambda^+_q - \Lambda^-_q}{2\sqrt{\omega_q}}, \tag{2.87}$$

$$G_q = \frac{1}{\sqrt{4\pi\omega_q}}, \qquad G_n = -\frac{\sin(|n|\pi r)}{\sqrt{\pi\omega_n}} \tag{2.88}$$

where

$$\Lambda^+_q = \sqrt{\omega_q - \beta_r\mu\alpha_3}, \qquad \Lambda^-_q = e(q)\sqrt{\omega_q + \beta_r\mu\alpha_3}, \tag{2.89}$$

$$\Lambda^+_n = \sqrt{\omega_n - \mu\alpha_3}, \qquad \Lambda^-_n = e(n)\sqrt{\omega_n + \mu\alpha_3}. \tag{2.90}$$

2.2.4 The contact interaction

Consider the commutation relation (2.64) at the next order in the coupling constant - i.e. $\mathcal{O}(g_2^2)$, we have

$$\{Q_{2\,\alpha_1\dot{\alpha}_2}, Q_{4\,\beta_1\dot{\beta}_2}\} + \{Q_{4\,\alpha_1\dot{\alpha}_2}, Q_{2\,\beta_1\dot{\beta}_2}\} + \{Q_{3\,\alpha_1\dot{\alpha}_2}, Q_{3\,\beta_1\dot{\beta}_2}\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H_4\,. \tag{2.91}$$

Determining Q_4 has been a long sought-after but yet to be realized undertaking since the early days of light-cone string field theory on flat space [47–50]. Since it remains to be determined, the solution in the plane-wave case has been to simply set it to zero. This is a self-consistent choice which gives rise to the so-called *contact interaction* (see appendix A)

$$H_4 = \frac{1}{4}Q_3^{\alpha_1\dot{\alpha}_2}Q_{3\,\alpha_1\dot{\alpha}_2}. \tag{2.92}$$

In flat space [47–50], the $2 \to 2$ string process requires a contribution from Q_4 to close the algebra (2.91). Here, we will be concerned with the plane-wave $1 \to 1$ string process;

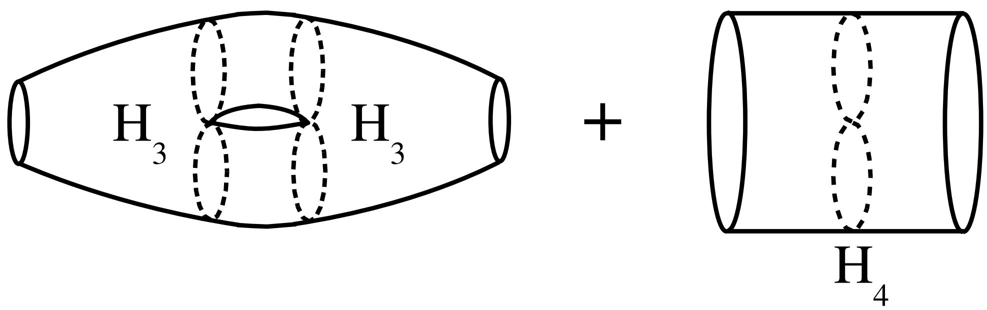


Figure 2.7: The one-loop process contributing to the shift of the energy or mass of a string state. Two H_3 vertices may be combined to form a standard one-loop diagram. The contact interaction H_4 (shown on the right) also contributes to this process. Unlike the first diagram, H_4 acts at a single time, while each H_3 acts at a different time; hence the name *contact*.

specifically the one-loop mass shift depicted in figure 2.7. Here two H_3 vertices alternately split and then rejoin the strings at separated light-cone times, while the contact interaction coalesces the splitting and joining to a single event (the moment of *contact*). It was argued in [81] that Q_4 cannot contribute to the $1 \to 1$ string process on account of it being quartic in string fields at tree level. Later, in analyzing the $1 \to 1$ string process on the plane-wave [86] argued less restrictively that although setting $Q_4 = 0$ in this setting still allows the algebra to close, this is only a necessary but not sufficient condition. It is the opinion of the author that this issue has not been fully resolved; however the work in this thesis follows the fashion of setting this quartic supercharge to zero only because of the lack of another option. Determining the full expression for Q_4 in the flat space or in the plane-wave background remains a potentially crucial element in the development of the light-cone string field theory.

2.2.5 One-loop mass shift: impurity conserving channel

We are now in a position to attempt the string theory calculation of the gauge theory result (2.15). The gauge theory result is valid for a general two-Φ^i-impurity operator (see (2.9), (2.12)), independent of the $SO(4) \times SO(4)$ representation (i.e. the spacetime index structure of the impurities). This allows for a choice of string state to consider for the calculation. Ideally, we would like to choose an $SO(4) \times SO(4)$ representation which can only be constructed out of bosonic oscillators. In this way, one circumvents having to worry about mixing between different string states of the same uncorrected energy. For example consider the representation which is scalar in both $SO(4)$'s

$$|[\mathbf{1},\mathbf{1}]\rangle = \begin{cases} \alpha_n^{\dagger k} \alpha_{-n}^{\dagger k} |\alpha\rangle \\ \beta^\dagger_{n\,\alpha_1\alpha_2} \beta^{\dagger\,\alpha_1\alpha_2}_{-n} |\alpha\rangle \end{cases} \tag{2.93}$$

i.e. it can either be constructed out of two fermionic, or two bosonic oscillators (or to mirror the gauge theory discussion "impurities"). These two states have the same energy at $g_2 = 0$, but when interactions are turned on, generically they will mix. To avoid this unpleasantness [86] used the following state[6]

$$|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} = \frac{1}{\sqrt{2}}\left(\alpha_n^{\dagger\, i}\alpha_{-n}^{\dagger\, j} + \alpha_n^{\dagger\, j}\alpha_{-n}^{\dagger\, i} - \frac{1}{2}\delta^{ij}\alpha_n^{\dagger\, k}\alpha_{-n}^{\dagger\, k}\right)|\alpha\rangle \tag{2.94}$$

whose $SO(4)\times SO(4)$ representation is unique. The one-loop mass shift proceeds using standard quantum mechanical perturbation theory

$$\delta E_n^{(2)} = \langle\phi_n|H_3\frac{\mathcal{P}}{E_n^{(0)} - H_2^{\rm int}}H_3|\phi_n\rangle + \langle\phi_n|H_4|\phi_n\rangle \tag{2.95}$$

where $|\phi_n\rangle$ represents the state whose shift we are calculating (i.e. (2.94)), which we take to be string #3 with uncorrected energy $E_n^{(0)}$, and where $\mathcal{P}$ is a projection operator on the space of two-string states. Finally $H_2^{\rm int}$ is the free Hamiltonian (2.37), restricted to the internal strings 1 and 2. In practice it is not feasible to consider the full range of intermediate two-string states; instead, a cue is taken from the gauge theory computation where the total number of impurities contained in intermediate states is equal to that of the external state[7]. In the string theory computation, this is the so-called impurity-conserving channel, which is realized as follows

$$(1+\delta^{ij})\,\delta E_n^{(2)} = {}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|H_3\frac{\mathbf{1}_B}{E_n^{(0)} - H_2^{\rm int}}H_3|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} + \frac{1}{4}{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|Q_3^\dagger\,\mathbf{1}_F\,Q_3|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} \tag{2.96}$$

where[8]

$$\begin{aligned}
\mathbf{1}_B =& \sum_{K,L=1}^{8}\int_0^1\frac{dr}{2\,r(1-r)}\Big(\sum_p \alpha_p^{\dagger\,K}\,\alpha_{-p}^{\dagger\,L}\,|\alpha_1\rangle\,|\alpha_2\rangle\langle\alpha_2|\,\langle\alpha_1|\,\alpha_{-p}^{L}\,\alpha_p^{K}\\
&\qquad + \alpha_0^{\dagger\,K}\,|\alpha_1\rangle\,\alpha_0^{\dagger\,L}\,|\alpha_2\rangle\langle\alpha_2|\,\alpha_0^{L}\langle\alpha_1|\,\alpha_0^{K}\Big)\\
\mathbf{1}_F =& \sum_{\Sigma_1,\Sigma_2}\int_0^1\frac{dr}{r(1-r)}\Big(\sum_p \alpha_p^{\dagger\,K}\,\beta_{-p}^{\dagger\,\Sigma_1\,\Sigma_2}\,|\alpha_1\rangle\,|\alpha_2\rangle\langle\alpha_2|\,\langle\alpha_1|\,\beta_{-p}^{\Sigma_1\,\Sigma_2}\,\alpha_p^{K}\\
&\qquad + \alpha_0^{\dagger\,K}\,|\alpha_1\rangle\,\beta_0^{\dagger\,\Sigma_1\,\Sigma_2}\,|\alpha_2\rangle\langle\alpha_2|\,\beta_0^{\Sigma_1\,\Sigma_2}\,\langle\alpha_1|\,\alpha_0^{K}\Big).
\end{aligned} \tag{2.97}$$

where $r\equiv-\alpha_1/\alpha_3$ so that $1-r=-\alpha_2/\alpha_3$ where we remind the reader that $\alpha_3<0$, while $\alpha_{1,2}>0$. The indices Σ_1 and Σ_2 are shorthand for indicating a sum over both dotted and un-dotted fermions. These projectors obey the condition $\mathbf{1}_{B,F}^2=\mathbf{1}_{B,F}$, where we note further that the vacuua are normalized by

[6]The normalization of this state is $1+\frac{1}{2}\delta^{ij}$. One could have equally chosen $|[\mathbf{1},\mathbf{9}]\rangle$; the string field theory would not produce a different result for the one-loop mass shift.

[7]There is no reason for this logic to be extended to the string theory picture. Indeed, the results of the next section show that it is an unjustified truncation.

[8]Oscillators act only on the vacuum closest to them.

$$\langle\alpha_1|\langle\alpha_2|\alpha_2\rangle|\alpha_1\rangle = r(1-r), \qquad \langle\alpha_3|\alpha_3\rangle = 1. \tag{2.98}$$

Note that strictly we should have added a two fermion state to $\mathbf{1}_B$. For the $|[\mathbf{9},\mathbf{1}]\rangle$ state however, this contributes nothing as it requires a trace of the i,j indices.

In order to calculate $\delta E_n^{(2)}$, we will require the following matrix elements [86]

$$\begin{aligned}
{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|\,\langle\alpha_2|\,\alpha_0^l\,\langle\alpha_1|\,\alpha_0^k\,|H_3\rangle &= -2\,r(1-r)\Big(\frac{\omega_{n(3)}}{\alpha_3}+\mu\Big)\,\widetilde{N}^{31}_{n,0}\,\widetilde{N}^{32}_{n,0}\,\Delta^{ijkl}\\
{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|\,\langle\alpha_2|\,\langle\alpha_1|\,\alpha_{-p}^l\,\alpha_p^k\,|H_3\rangle &= -2\,r(1-r)\Big(\frac{\omega_{n(3)}}{\alpha_3}-\frac{\omega_{p(1)}}{\alpha_3 r}\Big)\,\widetilde{N}^{31}_{n,p}\,\widetilde{N}^{31}_{n,-p}\,\Delta^{ijkl}
\end{aligned} \tag{2.99}$$

where we have used (C.10) and (C.11), and

$$\begin{aligned}
&{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|\,\langle\alpha_2|\,(\beta_0)^{\dot\sigma_1\dot\sigma_2}\,\langle\alpha_1|\,\alpha_0^k\,|Q_{3\,\beta_1\dot\beta_2}\rangle =\\
&\qquad -2i\,\bar{C}\,G_{0(2)}\,\big(K_{n(3)}+K_{-n(3)}\big)\,\widetilde{N}^{31}_{n,0}\,\Delta^{ijkl}(\sigma^l)_{\beta_1}^{\dot\sigma_1}\,\delta_{\dot\beta_2}^{\dot\sigma_2}\\
&{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|\,\langle\alpha_2|\,\langle\alpha_1|\,(\beta_{-p})^{\dot\sigma_1\dot\sigma_2}\,\alpha_p^k\,|Q_{3\,\beta_1\dot\beta_2}\rangle =\\
&\qquad -2i\,\bar{C}\,G_{|p|(1)}\,\big(K_{n(3)}\widetilde{N}^{31}_{n,p}+K_{-n(3)}\widetilde{N}^{31}_{n,-p}\big)\,\Delta^{ijkl}(\sigma^l)_{\beta_1}^{\dot\sigma_1}\,\delta_{\dot\beta_2}^{\dot\sigma_2}
\end{aligned} \tag{2.100}$$

where $\Delta^{ijkl} \equiv \frac{1}{\sqrt{2}}\big\{\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}-\frac{1}{2}\delta^{ij}\delta^{kl}\big\}$ and $\bar{C} \equiv \frac{\bar\eta}{4}\sqrt{-\frac{\alpha'}{2\,\alpha_3^3}}\,\sqrt{r(1-r)}$. The energy shift (2.96) is then calculated by taking the modulus squared of the matrix elements (2.99) and dividing by the energy denominator

$$\frac{-\alpha_3}{2\,(\omega_n - r^{-1}\omega_p)} \tag{2.101}$$

then adding $\frac{1}{4}$ of the modulus squared of the matrix elements (2.100), and then summing over the intermediate mode number p. Also note that the normalization $1+\frac{1}{2}\delta^{ij}$ results from

$$\sum_{k,l}\Delta^{ijkl}\Delta^{ijkl} = 1+\frac{1}{2}\delta^{ij}, \qquad \sum_{k,l}\sum_{\dot\sigma_1,\dot\sigma_2}\Big|\Delta^{ijkl}\,\big(\sigma^l\big)_{\beta_1}^{\dot\sigma_1}\,\delta_{\dot\beta_2}^{\dot\sigma_2}\Big|^2 = 4\left(1+\frac{1}{2}\delta^{ij}\right). \tag{2.102}$$

The forms of the summand in the sum over p may be massaged into two classes

$$F_1 = \sum_p \frac{P(p)}{Q(p)\sqrt{p^2+(r\mu\alpha_3)^2}}, \qquad F_2 = \sum_p \frac{P(p)}{Q(p)} \tag{2.103}$$

where $Q(p)$ and $P(p)$ are polynomials in p. In order to extract the large-μ behaviour of the sums, the contour integral method is employed

$$\sum_{p=-\infty}^{\infty} f(p) = -\frac{i}{2}\oint dz\, f(z)\,\cot(\pi z). \tag{2.104}$$

Rotating and scaling the integration variable through the substitution $z \to ixrz$, where $x = -\mu\alpha_3$, turns the cotangent into $\coth(\pi xrz)$ which can be set to one in the large x

limit[9]. If the summand $f(z)$ has no poles on the real axis, the procedure simply replaces p by $p' = rx\,p$ and integrates

$$\sum_{p=-\infty}^{\infty} f(p) = \int_{-\infty}^{\infty} dp' f(p') \tag{2.105}$$

yielding the large x behaviour. If there are poles on the real axis, one must evaluate their residue using the integrand in (2.104) and then integrate along any cut which $f(z)$ may possess along the imaginary axis. Thus we have

$$\begin{aligned} F_1 = -\pi \sum_i \mathrm{Res}\left(\cot(\pi p)\frac{P(p)}{Q(p)}, p_i \in \{p\,|\,Q(p)=0\}\right) \\ + \int_1^{\infty} dz \frac{[Q(ixrz)]^* P(ixrz) + \mathrm{c.c.}}{|Q(ixrz)|^2 \sqrt{z^2-1}} \end{aligned} \tag{2.106}$$

while for F_2 the second term is dropped as there is no cut. This calculation was originally presented in [86]. The author of this thesis finds error in the result reported there[10], as regards the $\lambda'^{3/2}$ and $\lambda'^{5/2}$ powers. A careful recalculation reveals the following result [90][11]

$$\boxed{\begin{aligned} \delta E_n^{(2)}/\mu = \frac{g_2^2}{4\pi^2}\Bigg[&\left(\frac{1}{24}+\frac{65}{64\pi^2 n^2}\right)\lambda' + \frac{3}{16}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} \\ &- n^2\left(\frac{1}{48}+\frac{89}{128\pi^2 n^2}\right)\lambda'^2 - \frac{9\,n^2}{32}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{5/2} \\ &+ n^4\left(\frac{1}{64}+\frac{339}{512\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{59}{160\pi^2}+\frac{45}{256\pi}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg] \end{aligned}} \tag{2.107}$$

where we note that the final integration over r in (2.97) is performed as the last step. The appearance of half-integer powers of λ' is disconcerting, as it is hard to see how such terms could ever arise in the gauge theory; they are clearly absent from (2.15). These terms appear to be generic to light-cone string field theory on the plane-wave [55] and so must find a way to cancel-out if a true matching to gauge theory is to be realized.

2.3 Divergence cancellation and impurity non-conserving channel

This section is a presentation of the author's original work published in arXiv:hep-th/0508126 [89]. In what follows, some passages are taken directly from that publication.

[9] The terms neglected by this approximation are of order $\exp(-\mu|\alpha_3|)$.

[10] The error is made in the evaluation of the sum using the contour-integral method.

[11] The undetermined function f in (2.68) is set to $r^{-1}(1-r)^{-1}$ here.

The result (2.107) does not match the gauge theory result (2.15) even at leading order. In an earlier attempt at this calculation [81], a reflection symmetry factor of $\frac{1}{2}$ was added in front of the H_3 term in (2.96), which the authors of [86] argued was incorrect. This factor produced a leading order agreement with gauge theory. The subleading orders were calculated by the author of this thesis [90]

$$\begin{aligned}\frac{1}{\mu}\delta E_n^{(2)\text{ref. symm.}} &= \frac{g_2^2}{4\pi^2}\left[\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\lambda' - \frac{n^2}{2}\lambda'^2\right) + \frac{n^2}{16\pi^2}\lambda'^{5/2}\right.\\ &+ \left. n^4\left(\frac{1}{32}+\frac{117}{256\pi^2 n^2}\right)\lambda'^3 - \frac{7n^4}{80\pi^2}\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\right] \end{aligned} \tag{2.108}$$

where it was found that the agreement persists up to $\mathcal{O}(\lambda'^2)$. Although this agreement is tantalizing, the factor of $\frac{1}{2}$ can not be justified, for reasons beyond the arguments of [86], who pointed out that (2.96) is standard quantum mechanical perturbation theory, not a field theory Feynman diagram prescription. The author's work [89] provided very convincing evidence in support of [86], which will be presented in this section.

The disagreement of (2.107) with gauge theory led [86] to conclude that (modulo the absence of Q_4) the truncation to the impurity conserving channel may be the source of the discrepancy. In fact, in the earlier work [81], a statement was made concerning the four-impurity channel. The claim was that the mode number sums diverge linearly if the large-μ limit is taken pre-summation; this means that if the sum is evaluated first, and then the large-μ limit taken (the method applied in section 2.2.5), this would result in a contribution to the mass shift which goes as $\sqrt{\lambda'}$. The idea is that (roughly)

$$\begin{aligned}&\sum_p \frac{1}{p^2+\mu^2} \rightarrow \text{large-}\mu \text{ limit} \rightarrow \frac{1}{\mu^2}\sum_p 1 \sim \lambda'\cdot\infty\\ &\sum_p \frac{1}{p^2+\mu^2} \simeq \frac{1}{\mu}\int dx \frac{1}{x^2+1} \sim \sqrt{\lambda'}.\end{aligned} \tag{2.109}$$

The prediction was therefore that the four-impurity channel should give a contribution larger than the impurity-conserving channel in the large-μ limit. Further, the $\sqrt{\lambda'}$ indicated a non-perturbative origin in the gauge theory. In the paper [89], the author of this thesis and his collaborators undertook a proper investigation of the four-impurity channel (while making arguments concerning higher impurity channels) to verify the claim of [81]. What was discovered was that $\sqrt{\lambda'}$ behaviour is a reflection of real (logarithmic) divergences in the H_3 and contact amplitudes which cancel, taking with them the $\sqrt{\lambda'}$ terms. Finiteness and the perturbative nature of the mass-shift were thus established in concert. The analysis revealed further that generically, every order in intermediate state impurities contributes a leading λ' contribution to the mass shift; a discouraging result as regards matching to the gauge theory.

2.3.1 Invitation: trace state

The logarithmic divergences found in the four impurity channel are at play in a simpler setting. A careful calculation of the impurity-conserving channel contribution to the mass shift of the normalized bosonic trace state

$$|[\mathbf{1},\mathbf{1}]\rangle = \frac{1}{2}\alpha_n^{i\dagger}\alpha_{-n}^{i\dagger}|\alpha\rangle \tag{2.110}$$

reveals the same divergences and cancellation mechanism. In [82], this calculation was performed by taking the large μ limit first, then summing over mode numbers. That procedure found a finite result. However, if μ is kept finite, there are logarithmically divergent summations which must be dealt with before the large μ limit is taken. The H_3 matrix element contributing to the mass shift is

$$\begin{aligned}\langle\alpha_3|\frac{1}{2}\alpha_n^i\alpha_{-n}^i\,\langle\alpha_2|\langle\alpha_1|\alpha_p^K\alpha_{-p}^L|H_3\rangle = -g_2\,\frac{r\,(1-r)}{8}&\left[8\left(\frac{\omega_n^{(3)}}{\alpha_3}+\frac{\omega_p^{(1)}}{\alpha_1}\right)\widetilde{N}^{3\,1}_{-n\,p}\widetilde{N}^{3\,1}_{n\,p}\,\delta^{kl}\right.\\ &\left.+16\,\frac{\omega_n^{(3)}}{\alpha_3}\widetilde{N}^{3\,3}_{n\,n}\widetilde{N}^{1\,1}_{p\,-p}\,\delta^{KL}+16\,\frac{\omega_p^{(1)}}{\alpha_1}\widetilde{N}^{3\,3}_{n\,-n}\widetilde{N}^{1\,1}_{p\,p}\,\Pi^{KL}\right]\end{aligned} \tag{2.111}$$

where the index $i=1,\ldots,4$ is summed over. Note that $K,L=1,\ldots,8$, while δ^{kl} is non-zero only for $k=l=1,\ldots,4$. The matrix Π^{KL} is given by

$$\Pi^{KL} = \mathrm{diag}(1,1,1,1,-1,-1,-1,-1). \tag{2.112}$$

When calculating the H_3 contribution to the mass shift it is only the very last term in (2.111) which is divergent. Singling-out its contribution, one finds (using the projectors (2.97))

$$\delta E^{\mathrm{div}}_{H_3} = \int_0^1 dr\,\left(g_2\frac{r(1-r)}{8}\right)^2\frac{-\alpha_3}{2\,r\,(1-r)}\sum_{KL}\sum_{p=-\infty}^{\infty}\frac{\left[16\,\frac{\omega_p}{-r\,\alpha_3}\widetilde{N}^{3\,3}_{n\,-n}\,\widetilde{N}^{1\,1}_{p\,p}\,\Pi^{KL}\right]^2}{2\,\omega_n-2\,r^{-1}\omega_p} \tag{2.113}$$

Inspection of the forms of the Neumann matrices (see appendix C) reveal that the numerator in (2.113) goes like a constant for large $|p|$, and thus the sum as a whole goes like $1/|p|$ for $|p|\gg|\mu\alpha_3|$. This is a logarithmically diverging sum. In [82] the strict large μ limit was taken for the energy denominator, leading to a convergent $1/p^2$ behaviour instead. Here we will stick with the finite μ expressions and show that the divergence is removed by the contact term. Note that a double fermionic impurity intermediate state also contributes to the H_3 piece, however it does not display any divergent behaviour. In appendix D, section D.5, the contribution from this channel is calculated, as an example of how these calculations are performed in general. Further, the $\alpha_0^\dagger|\alpha_1\rangle\alpha_0^\dagger|\alpha_2\rangle$ intermediate state is unimportant as it does not contain a mode number sum. The contribution from the contact term stems from the following matrix element

$$\begin{aligned}&\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1}\langle\alpha_3|\frac{1}{2}\alpha_n^i\alpha_{-n}^i\,\langle\alpha_2|\langle\alpha_1|\alpha_p^K\beta_{-p}^{\Sigma_1\,\Sigma_2}|Q_{3\,\beta_1\dot\beta_2}\rangle\\ &\quad=\left(G^{(1)}_{|p|}\,K^{(3)}_n\widetilde{N}^{3\,1}_{n\,p}+G^{(1)}_{|p|}\,K^{(3)}_{-n}\widetilde{N}^{3\,1}_{-n\,p}\right)(\sigma^k)^{\dot\sigma_1}_{\beta_1}\delta^{\dot\sigma_2}_{\dot\beta_2}+4\,G^{(1)}_{|p|}\,K^{(1)}_{-p}\widetilde{N}^{3\,3}_{n\,-n}(\sigma^K)^{\Sigma}_{\beta}\delta^{\Sigma}_{\beta}\end{aligned} \tag{2.114}$$

Here $K = 1, \ldots, 8$ while the Σ and β indices are either dotted or undotted as required by the particular SO(4) representation indicated by K. The last term in (2.114) gives rise to a log-divergent sum. For large positive p, $(K^{(1)}_{-p})^2$ goes as a constant, and so the sum is controlled by $(G^{(1)}_{|p|})^2$ which goes as $1/p$, and hence diverges logarithmically. For p negative, the sum converges. Thus, the divergent contribution to $\delta E^{(2)}$ is found to be (again using (2.97))

$$\delta E^{\text{div}}_{H_4} = 8 \int_0^1 dr \left(g_2 \frac{1}{4} \sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}} \right)^2 \frac{1}{r(1-r)} \sum_{p=1}^{\infty} \left(4\, G^{(1)}_{|p|}\, K^{(1)}_{-p} \widetilde{N}^{3\,3}_{n\,-n} \right)^2 \tag{2.115}$$

where the leading factor of 8 comes from the sum over K. Again the intermediate state $\alpha_0^\dagger|\alpha_1\rangle\beta_0^\dagger|\alpha_2\rangle$ is unimportant to convergence and is ignored here. In taking the large p limits of the summands in (2.113) and (2.115), one finds,

$$\delta E^{\text{div}}_{H_3} \sim -\frac{1}{2} \int_0^1 dr\, \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2} \left(\widetilde{N}^{3\,3}_{n\,-n} \right)^2 \frac{1}{|p|}, \tag{2.116}$$

$$\delta E^{\text{div}}_{H_4} \sim + \int_0^1 dr\, \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2} \left(\widetilde{N}^{3\,3}_{n\,-n} \right)^2 \frac{1}{p}. \tag{2.117}$$

Noting that in the H_3 contribution the divergence is found for both positive and negative p, while in the H_4 contribution the divergence occurs only for positive p, and hence a relative factor of 2 is induced in the H_3 term, one sees that the logarithmically divergent sums cancel identically between the H_3 and contact terms, leaving a convergent sum. As promised at the beginning of section 2.3, this cancellation fixes the relative weight of the H_3 and contact terms to that employed in [86]. It contradicts the reflection symmetry factor of 1/2 originally given in [81]; finiteness of the string theory amplitude requires the absence of this factor.

2.3.2 Four impurity channel

We now consider the mass shift of the $\big|[\mathbf{9},\mathbf{1}]\big\rangle^{(ij)}$ string state (2.94) due to intermediate states which contain four impurities. In the explicit expression for the matrix element to be quoted below, we shall see that the parameter $\mu\alpha_3$ occurs only in combinations involving ω_p and there is a duality between the large p and the large $\mu\alpha_3$ limits. Therefore, since a logarithmic divergence in the sums indicates that the summands have as many (inverse) powers of the summation variables as there are summation variables, this translates into vanishing $\mu\alpha_3$ dependence for this contribution to $\delta E^{(2)}$, leaving $\delta E^{(2)}/\mu \sim \sqrt{\lambda'}$. It is thus seen that $\sqrt{\lambda'}$ behaviour is simply the result of log divergences, which should, if pp-wave light-cone string field theory is to make any sense, cancel out entirely. We begin with the H_3 contribution to the mass shift. We consider the following intermediate state[12]

[12]From now on, the sum over intermediate state spacetime indices is implied, rather than explicitly indicated.

$$\mathbf{1}_B = \int_0^1 \frac{dr}{4!\, r(1-r)} \sum_{p_1\, p_2\, p_3\, p_4} \alpha^{\dagger\, K}_{p_1}\, \alpha^{\dagger\, L}_{p_2}\, \alpha^{\dagger\, M}_{p_3}\, \alpha^{\dagger\, N}_{p_4}\, |\alpha_1\rangle\, |\alpha_2\rangle\langle\alpha_2|\, \langle\alpha_1|\, \alpha^N_{p_4}\, \alpha^M_{p_3}\, \alpha^L_{p_2}\, \alpha^K_{p_1} \tag{2.118}$$

where the sum over mode numbers is restricted by the level matching condition $\sum_i p_i = 0$. Although there are many possible contractions of this state with the oscillators in $|H_3\rangle$, we will only be concerned with those which lead to log divergent sums. These are the ones where both oscillators in the prefactor of $|H_3\rangle$ contract with the oscillators in $\mathbf{1}_B$. We find this contribution to $\delta E^{(2)}$ to be

$$\begin{aligned} \delta E^{\mathrm{div}}_{H_3} = \int_0^1 \frac{dr}{4!\, r\,(1-r)} \left(g_2 \frac{r(1-r)}{4}\right)^2 \sum_{p_2\, p_3\, p_4} \frac{-\alpha_3\, r}{2\,\omega_n\, r - \sum_{i=1}^4 \omega_{p_i}} \times \\ \left(2\, \frac{\omega_{p_1}+\omega_{p_2}}{-r\,\alpha_3}\, \widetilde{N}^{1\,1}_{-p_1\, p_2}\right)^2 8\cdot 12 \left\{ \left(\widetilde{N}^{3\,1}_{n\, p_3} \widetilde{N}^{3\,1}_{-n\, p_4}\right)^2 + \widetilde{N}^{3\,1}_{n\, p_3} \widetilde{N}^{3\,1}_{-n\, p_3} \widetilde{N}^{3\,1}_{n\, p_4} \widetilde{N}^{3\,1}_{-n\, p_4} \right\} \end{aligned} \tag{2.119}$$

where $p_1 = -(p_2 + p_3 + p_4)$. The factor of 12 is combinatoric and counts the number of ways equivalent contractions can be made. The factor of 8 comes from a sum over the spacetime indices of $\mathbf{1}_B$. It is easy to see that in the above, the sum over p_2 is log divergent. In fact, it is the very same form as appears in (2.113). In order to evaluate the leading μ dependence of the expression (2.119), one must consider the forms of the Neumann matrices given in appendix C. The matrices which have one leg in the external string, i.e. $\widetilde{N}^{3\,r}_{n\,p}$ and $\widehat{Q}^{3\,r}_{n\,p}$, contain poles at $p = \beta_r n$. The sums over mode numbers involved with these Neumann matrices are then dominated by the residues given in (2.106). Specifically, they are $\mathcal{O}(\mu^0)$. This allows us to dispense with the sums over p_3 and p_4 in (2.119), as far as μ power counting is concerned. The remaining sum over p_2 is executed via replacing p_2 with $z = \mu|\alpha_3|p'$ and integrating over p'. One then finds that the μ dependence drops out completely from the squared term involving $\widetilde{N}^{1\,1}_{-p_1\, p_2}$, while the measure of the integration over p' cancels the μ^{-1} stemming from the energy denominator. One then has that $\delta E^{\mathrm{div}}_{H_3} \sim$ constant, and therefore $\delta E^{(2)}/\mu \sim \sqrt{\lambda'}$. There are also contributions from intermediate states which contain two bosonic and two fermionic impurities, however these produce convergent sums and $\mathcal{O}(\lambda')$ contributions to $\delta E^{(2)}/\mu$. The four-fermion channel is forbidden because it produces a delta function (i.e. a trace) on the external state's spacetime indices.

We now show that the contact term contribution stemming from the following intermediate state,

$$\mathbf{1}_F = \int_0^1 \frac{dr}{3!\, r(1-r)} \sum_{p_1\, p_2\, p_3\, p_4} \beta^{\dagger\, \Sigma_1\Sigma_2}_{p_1}\, \alpha^{\dagger\, L}_{p_2}\, \alpha^{\dagger\, M}_{p_3}\, \alpha^{\dagger\, N}_{p_4}\, |\alpha_1\rangle\, |\alpha_2\rangle\langle\alpha_2|\, \langle\alpha_1|\, \alpha^N_{p_4}\, \alpha^M_{p_3}\, \alpha^L_{p_2}\, \beta^{\Sigma_1\Sigma_2}_{p_1} \tag{2.120}$$

cancels the divergent piece coming from the H_3 contribution, leaving an $\mathcal{O}(\lambda')$ contribution to $\delta E^{(2)}/\mu$. The log divergent piece comes from contractions where the $\alpha^\dagger$ in the prefactor of $|Q_3\rangle$ is joined with one of the bosonic oscillators in $\mathbf{1}_F$. One finds,

$$\delta E^{\mathrm{div}}_{H_4} = \int_0^1 \frac{dr}{3!\,r\,(1-r)} \left(g_2 \frac{1}{4} \sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}} \right)^2 \sum_{p_2\,p_3\,p_4} (2\,G_{p_1} K_{-p_2})^2$$
$$\times 8 \cdot 6 \left\{ \left(\widetilde{N}^{3\,1}_{n\,p_3} \widetilde{N}^{3\,1}_{-n\,p_4} \right)^2 + \widetilde{N}^{3\,1}_{n\,p_3} \widetilde{N}^{3\,1}_{-n\,p_3} \widetilde{N}^{3\,1}_{n\,p_4} \widetilde{N}^{3\,1}_{-n\,p_4} \right\} \tag{2.121}$$

In the above one sees the very same pattern as was seen in section 2.3.1 for the trace state. The sum over p_2 is divergent on the positive side, and cancels the divergence in (2.119).

The remaining (convergent) expression gives an $\mathcal{O}(\lambda')$ contribution to $\delta E^{(2)}/\mu$. To see this we note that the poles in p_3 and p_4 set these variables to an $\mathcal{O}(\mu^0)$ quantity. Then $p1 = \epsilon - p_2$ by level matching, where $\epsilon = 2\,r\,n$ represents this $\mathcal{O}(\mu^0)$ quantity. Ignoring the common factor involving the $\widetilde{N}^{3\,1}_{n\,p_i}$'s, the remaining expressions may be expressed as (suppressing the integration over r)

$$\delta E^{\mathrm{div}}_{H_3} + \delta E^{\mathrm{div}}_{H_4} = g_2^2 \frac{(1-r)}{(4\pi)^2 |\alpha_3|} \left[\frac{2(\omega_2 + p_2)}{\omega_1 \omega_2} + \frac{2(\omega_1\omega_2 + r^2\mu^2\alpha_3^2 - p_1 p_2)}{(2\,r\,\omega_n - \omega_1 - \omega_2 - \omega_3 - \omega_4)\,\omega_1\omega_2} \right]. \tag{2.122}$$

The next step is to notice that $2\,r\,\omega_n - \omega_3 - \omega_4 = 0$ in the large-μ limit due to p_3 and p_4 being set to $r\,n$. Combining the terms, the leading large-μ behaviour is given by

$$\delta E^{\mathrm{div}}_{H_3} + \delta E^{\mathrm{div}}_{H_4} = g_2^2 \frac{(1-r)}{(4\pi)^2 |\alpha_3|} 2 \sum_{p_2=-\infty}^{\infty} \frac{p_2}{\sqrt{p_2^2 + (r\mu\alpha_3)^2}\sqrt{(p_2-\epsilon)^2 + (r\mu\alpha_3)^2}} \sim \mu^{-1} \tag{2.123}$$

which results in the advertised large-μ behaviour. Again, there is a non-divergent contribution from the intermediate state with three fermionic and one bosonic impurity which we will ignore. There are two other choices for distributing the intermediate-state oscillators amongst the two strings. We may express them in pairs

$$\mathbf{1}_B = \int_0^1 \frac{dr}{3!\,r(1-r)} \sum_{p_1\,p_2\,p_3} \alpha^{\dagger\,K}_{p_1}\, \alpha^{\dagger\,L}_{p_2}\, \alpha^{\dagger\,M}_{p_3}\, |\alpha_1\rangle\, \alpha^{\dagger\,N}_0\, |\alpha_2\rangle\langle\alpha_2|\, \alpha^N_0\, \langle\alpha_1|\, \alpha^M_{p_3}\, \alpha^L_{p_2}\, \alpha^K_{p_1}$$
$$\mathbf{1}_F = \int_0^1 \frac{dr}{2!\,r(1-r)} \sum_{p_1\,p_2\,p_3} \beta^{\dagger\,a}_{p_1}\, \alpha^{\dagger\,L}_{p_2}\, \alpha^{\dagger\,M}_{p_3}\, |\alpha_1\rangle\, \alpha^{\dagger\,N}_0\, |\alpha_2\rangle\langle\alpha_2|\, \alpha^N_0\, \langle\alpha_1|\, \alpha^M_{p_3}\, \alpha^L_{p_2}\, \beta^a_{p_1} \tag{2.124}$$

where $\sum_{i=1}^3 p_i = 0$ and,

$$\mathbf{1}_B = \int_0^1 \frac{dr}{2\cdot(2!)^2\,r(1-r)} \sum_{p_1\,p_2} \alpha^{\dagger\,K}_{p_1}\, \alpha^{\dagger\,L}_{-p_1}\, |\alpha_1\rangle\, \alpha^{\dagger\,M}_{p_2}\, \alpha^{\dagger\,N}_{-p_2}\, |\alpha_2\rangle\langle\alpha_2|\, \alpha^N_{-p_2}\, \alpha^M_{p_2}\, \langle\alpha_1|\, \alpha^L_{-p_1}\, \alpha^K_{p_1}$$
$$\mathbf{1}_F = \int_0^1 \frac{dr}{2!\,r(1-r)} \sum_{p_1\,p_2} \alpha^{\dagger\,K}_{p_1}\, \alpha^{\dagger\,L}_{-p_1}\, |\alpha_1\rangle\, \alpha^{\dagger\,M}_{p_2}\, \beta^{\dagger\,a}_{-p_2}\, |\alpha_2\rangle\langle\alpha_2|\, \beta^a_{-p_2}\, \alpha^M_{p_2}\, \langle\alpha_1|\, \alpha^L_{-p_1}\, \alpha^K_{p_1}. \tag{2.125}$$

One may show the very same cancellation mechanism applies between each pair here; essentially the difference is that the $\widetilde{N}^{3\,r}_{n\,p}$ factors in (2.119) and (2.121) have opposite string label r to the remaining factors in those expressions. Finally one may show that the intermediate state with a $\beta_0^\dagger$ alone on string 2 does not lead to a divergent contribution.

We therefore find that the entire contribution to $\delta E^{(2)}/\mu$ from the four impurity channel is convergent / leads as λ'. It is not hard to generalize the above argument to $\mathbf{1}_B$'s containing an arbitrary number of bosonic impurities and no fermionic impurities. The divergent expressions cancel against contact interactions with $\mathbf{1}_F$'s containing one fermionic and the same number (less-one) of bosonic oscillators as $\mathbf{1}_B$. Adding fermionic impurities is far less trivial because of the complicated nature of the prefactors of $|H_3\rangle$ and $|Q_3\rangle$. However, in the next section a more elegant argument is presented which claims the absence of log divergences for arbitrary impurity intermediate states.

It is important to note that the $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ state receives no contributions to its energy shift from the zero impurity channel and so we don't need to worry about $\sqrt{\lambda'}$ behaviour hiding there.

2.3.3 Generalizing to arbitrary impurities

It is possible to formally manipulate the contact term in such a way that the H_3 portion of the energy shift is cancelled entirely, leaving a convergent expression, which appears devoid of any $\sqrt{\lambda'}$ contributions to $\delta E^{(2)}/\mu$. The manipulation proceeds through the supersymmetry algebra; for completeness we include both the "dot-undot" and "undot-dot" representations of the supercharges, see appendix A. At order g_2 we have

$$\begin{aligned}\left\{Q_{2\alpha_1\dot\alpha_2},Q_{3\beta_1\dot\beta_2}\right\}+\left\{Q_{3\alpha_1\dot\alpha_2},Q_{2\beta_1\dot\beta_2}\right\}&=-2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}H_3\ ,\\ \left\{Q_{2\dot\alpha_1\alpha_2},Q_{3\dot\beta_1\beta_2}\right\}+\left\{Q_{3\dot\alpha_1\alpha_2},Q_{2\dot\beta_1\beta_2}\right\}&=-2\epsilon_{\dot\alpha_1\dot\beta_1}\epsilon_{\alpha_2\beta_2}H_3\end{aligned}\tag{2.126}$$

analogously to order g_2^2 one has

$$\begin{aligned}\left\{Q_{3\alpha_1\dot\alpha_2},Q_{3\beta_1\dot\beta_2}\right\}+\left\{Q_{2\alpha_1\dot\alpha_2},Q_{4\beta_1\dot\beta_2}\right\}+\left\{Q_{4\alpha_1\dot\alpha_2},Q_{2\beta_1\dot\beta_2}\right\}&=-2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}H_4,\\ \left\{Q_{3\dot\alpha_1\alpha_2},Q_{3\dot\beta_1\beta_2}\right\}+\left\{Q_{2\dot\alpha_1\alpha_2},Q_{4\dot\beta_1\beta_2}\right\}+\left\{Q_{4\dot\alpha_1\alpha_2},Q_{2\dot\beta_1\beta_2}\right\}&=-2\epsilon_{\dot\alpha_1\dot\beta_1}\epsilon_{\alpha_2\beta_2}H_4.\end{aligned}\tag{2.127}$$

In order to dispense with the $\epsilon_{\alpha\beta}$'s, we employ the Hermitian conjugates of Q_3, see again appendix A

$$\left\{Q_{2\beta_1\dot\beta_2},Q_3^{\beta_1\dot\beta_2}\right\}=+4H_3,\quad\left\{Q_{2\dot\beta_1\beta_2},Q_3^{\dot\beta_1\beta_2}\right\}=+4H_3\tag{2.128}$$

and

$$\begin{aligned}H_4=\frac{1}{8}Q_{3\beta_1\dot\beta_2}Q_3^{\beta_1\dot\beta_2}+\frac{1}{8}Q_{3\dot\beta_1\beta_2}Q_3^{\dot\beta_1\beta_2}+\frac{1}{8}Q_{4\beta_1\dot\beta_2}Q_2^{\beta_1\dot\beta_2}+\frac{1}{8}Q_{4\dot\beta_1\beta_2}Q_2^{\dot\beta_1\beta_2}&\\ +\frac{1}{8}Q_{2\beta_1\dot\beta_2}Q_4^{\beta_1\dot\beta_2}+\frac{1}{8}Q_{2\dot\beta_1\beta_2}Q_4^{\dot\beta_1\beta_2}.&\end{aligned}\tag{2.129}$$

Using these formula, the contribution of H_4 to $\delta E^{(2)}$ can be rewritten as a sum of a term which cancels the H_3 contribution plus other pieces which all contain Q_2 acting on one of the external states. Taking the expectation value of part of (2.129), and introducing P as a representation of unity, we have

$$\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2}+Q_{3\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2}\right\rangle=\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}P\frac{E_0-H_2}{E_0-H_2}Q_3^{\beta_1\dot{\beta}_2}\right\rangle \tag{2.130}$$

$$+\frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}P\frac{E_0-H_2}{E_0-H_2}Q_3^{\dot{\beta}_1\beta_2}\right\rangle. \tag{2.131}$$

It could be that the energy denominator which we have introduced here will have a zero. In that case, the projector P is a reminder to define the singularity using a principle value prescription[13]. Equation (2.130) can be written as

$$=-\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}\left[H_2,Q_3^{\beta_1\dot{\beta}_2}\right]\right\rangle-\frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}\left[H_2,Q_3^{\dot{\beta}_1\beta_2}\right]\right\rangle \tag{2.134}$$

where we remind the reader of (2.61), which ensures that H_2 acting on the external state gives a positive E_0. Up to order g_2 the following equation holds

$$\left[H_2,Q_3^{\beta_1\dot{\beta}_2}\right]=\left[Q_2^{\beta_1\dot{\beta}_2},H_3\right] \tag{2.135}$$

so that (2.134) becomes

$$=\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}\left[H_3,Q_2^{\beta_1\dot{\beta}_2}\right]\right\rangle+\frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}\left[H_3,Q_2^{\dot{\beta}_1\beta_2}\right]\right\rangle. \tag{2.136}$$

Since Q_2 commutes with H_2 one has

$$=\ +\ \frac{1}{8}\left\langle Q_{2\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}H_3\right\rangle+\frac{1}{8}\left\langle Q_{2\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}H_3\right\rangle$$

[13]There is one additional subtlety, the intermediate states must each obey the level-matching condition. This condition can be enforced by inserting a projection operator. For example, for two-string intermediate states, we can combine such a projector with the energy denominator as

$$\frac{P}{E_0-H_2}=\int_0^\infty d\tau\ e^{E_0\tau}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ e^{-H_2^{(1)}\tau+i\theta_1N^{(1)}}\ e^{-H_2^{(2)}\tau+i\theta_2N^{(2)}} \tag{2.132}$$

where

$$N^{(r)}=\sum_n n\left(a_n^{I(r)\dagger}a_n^{I(r)}+b_{an}^{(r)\dagger}b_{an}^{(r)}\right) \tag{2.133}$$

with $r=1,2$ are the level number operators for the two intermediate strings. The net effect of the operators in the above equation is to make the replacement $\left(a_n^{(r)\dagger},b_n^{(r)\dagger}\right)\to\left(e^{-\omega_n\tau+in\theta_{(r)}}a_n^{(r)\dagger},e^{-\omega_n\tau+in\theta_{(r)}}b_n^{(r)\dagger}\right)$ for all creation operators which lie to the right of the projector. Then, after the matrix element is computed, we multiply it by $e^{E_0\tau}$ and integrate over τ and θ_r. Any potential divergences come from the region near $\tau=0$.

$$
\begin{aligned}
&+ \frac{1}{8}\left\langle Q_{3\beta_1\dot\beta_2}\frac{P}{E_0-H_2}H_3Q_2^{\beta_1\dot\beta_2}\right\rangle + \frac{1}{8}\left\langle Q_{3\dot\beta_1\beta_2}\frac{P}{E_0-H_2}H_3Q_2^{\dot\beta_1\beta_2}\right\rangle \\
&- \left\langle H_3\frac{P}{E_0-H_2}H_3\right\rangle
\end{aligned} \tag{2.137}
$$

and the last term cancels the H_3 contribution to the energy shift. The final expression for the energy shift is

$$
\begin{aligned}
\delta E^{(2)} = &+ \frac{1}{8}\left\langle Q_{2\beta_1\dot\beta_2}Q_3^{\beta_1\dot\beta_2}\frac{P}{E_0-H_2}H_3\right\rangle + \frac{1}{8}\left\langle Q_{2\dot\beta_1\beta_2}Q_3^{\dot\beta_1\beta_2}\frac{P}{E_0-H_2}H_3\right\rangle \\
&+ \frac{1}{8}\left\langle Q_{3\beta_1\dot\beta_2}\frac{P}{E_0-H_2}H_3Q_2^{\beta_1\dot\beta_2}\right\rangle + \frac{1}{8}\left\langle Q_{3\dot\beta_1\beta_2}\frac{P}{E_0-H_2}H_3Q_2^{\dot\beta_1\beta_2}\right\rangle \\
&+ \frac{1}{4}\left\langle Q_{2\beta_1\dot\beta_2}Q_4^{\beta_1\dot\beta_2}\right\rangle + \frac{1}{4}\left\langle Q_{2\dot\beta_1\beta_2}Q_4^{\dot\beta_1\beta_2}\right\rangle \\
&+ \frac{1}{4}\left\langle Q_{4\beta_1\dot\beta_2}Q_2^{\beta_1\dot\beta_2}\right\rangle + \frac{1}{4}\left\langle Q_{4\dot\beta_1\beta_2}Q_2^{\dot\beta_1\beta_2}\right\rangle .
\end{aligned} \tag{2.138}
$$

It is amusing to note that the vanishing energy correction for a supersymmetric external state is manifest in (2.138), since if Q_2 annihilates the external state, all of the terms are identically zero. As was mentioned in section 2.2.4, Q_4 is unknown and it is consistent with the closure of the super-algebra to set it to zero here. Further, for the calculations at hand here, the "dot-undot" terms are identical to the "undot-dot" terms and so we continue to simply use double the latter.

Using the $\big|[\mathbf{9},\mathbf{1}]\big\rangle^{(ij)}$ external state, we can check that what is left is manifestly convergent for the four impurity channel, and then show that the addition of impurities will not disturb this, leaving $\mathcal{O}(\lambda')$ contributions at every order in impurities. We have two sorts of terms in (2.138), which we can represent schematically as follows

$$
\delta E_1 = \sum_I \frac{\Big(\langle\Phi|\langle I|Q_3\rangle\Big)\Big(\langle\Psi|\langle I|H_3\rangle\Big)^*}{E_\Phi - E_I} \qquad \delta E_2 = \sum_I \frac{\Big(\langle\Phi|\langle I|H_3\rangle\Big)\Big(\langle\Psi|\langle I|Q_3\rangle\Big)^*}{E_\Phi - E_I} \tag{2.139}
$$

where $|\Phi\rangle$ is the $\big|[\mathbf{9},\mathbf{1}]\big\rangle^{(ij)}$ external state, $|\Psi\rangle = Q_2|\Phi\rangle$, and $|I\rangle$ is a level-matched, two-string intermediate state. In order to evaluate the convergence and large μ behaviour of these terms, we can be entirely schematic. We take (see (A.22) for the expression of Q_2 in the BMN basis)

$$
|\Psi\rangle \sim \sqrt{-\mu\alpha_3}\,\beta_n^\dagger\,\alpha_{-n}^\dagger|\alpha_3\rangle \qquad |\Phi\rangle \sim \alpha_n^\dagger\,\alpha_{-n}^\dagger|\alpha_3\rangle \tag{2.140}
$$

while for the purpose of evaluating convergence we can take

$$
G_p^{(1)} \sim \frac{1}{\sqrt{p}} \qquad K_{-p}^{(1)} \sim \text{constant} \qquad \widetilde{N}^{3\,r}_{n\,p} \sim \frac{1}{p} \qquad \widetilde{N}^{r\,s}_{q\,p} \sim \frac{1}{p+q} \tag{2.141}
$$

where we take all integers to be positive. Let us begin with δE_1 in (2.139), we have two choices for four impurity intermediate states

$$
\begin{aligned}
|I\rangle &\sim \alpha^\dagger_{p_1}\beta^\dagger_{p_2}\alpha^\dagger_{p_3}\alpha^\dagger_{p_4}|\alpha_1\rangle|\alpha_2\rangle \\
|I\rangle &\sim \alpha^\dagger_{p_1}\beta^\dagger_{p_2}\beta^\dagger_{p_3}\beta^\dagger_{p_4}|\alpha_1\rangle|\alpha_2\rangle .
\end{aligned}
\tag{2.142}
$$

We can proceed with the first one, which will give

$$
\begin{aligned}
\delta E_1 \sim \sqrt{x} \sum_{p_1\, p_2\, p_3\, p_4} & \frac{1}{2\, r\, \omega_n - \sum_{i=1}^4 \,\omega_{p_i}} \langle\alpha_3|\alpha_n\,\alpha_{-n}\langle\alpha_2|\langle\alpha_1|\alpha_{p_1}\,\beta_{p_2}\,\alpha_{p_3}\,\alpha_{p_4}\,|Q_3\rangle \\
& \times\Big(\langle\alpha_3|\beta_n\,\alpha_{-n}\langle\alpha_2|\langle\alpha_1|\alpha_{p_1}\,\beta_{p_2}\,\alpha_{p_3}\,\alpha_{p_4}\,|H_3\rangle\Big)^*
\end{aligned}
\tag{2.143}
$$

where $x = -\mu\alpha_3$ and $\sum_i p_i = 0$. There are two general ways in which we can contract the $\beta^{(r)}$'s. They can connect to factors of $\sum_m G_m \beta^\dagger_m$ in the prefactors of $|H_3\rangle$ and $|Q_3\rangle$, or they can pair-up to bring down a factor of $\widehat{Q}^{r\,s}_{m\,p}$ from the exponential. As far as convergence and large x power-counting is concerned however, $G^{(r)}_m G^{(s)}_p$ is equivalent to $\widehat{Q}^{r\,s}_{m\,p}$, and so we will simply use the former. When contracting $\beta^{(3)}$'s there is a fundamental difference between $G^{(3)}_n G^{(r)}_p$ and $\widehat{Q}^{3\,r}_{n\,p}$, as far as large x behaviour is concerned, because of the pole in the latter. In fact $\widehat{Q}^{3\,r}_{n\,p}$ is essentially equivalent to $\widetilde{N}^{3\,r}_{n\,p}$ and therefore the two can be interchanged in this analysis.

Because K_{-p} goes as a constant for large p, the worst convergence will always be realized by contracting the intermediate bosonic impurities with the prefactors of $|H_3\rangle$ and $|Q_3\rangle$. These contractions will yield[14]

$$
\delta E_1 \sim \sqrt{x} \sum_{p_1\, p_2\, p_3\, p_4} \frac{G^{(1)}_{p_2}\widetilde{N}^{3\,1}_{-n\,p_1}K^{(1)}_{-p_3}\widetilde{N}^{3\,1}_{n\,p_4} \times K^{(1)}_{-p_3}K^{(1)}_{p_4}\widetilde{N}^{3\,1}_{-n\,p_1}\begin{cases}\widetilde{Q}^{3\,1}_{n\,p_2} - \widetilde{Q}^{1\,3}_{p_2\,n} \\ G^{(3)}_n G^{(1)}_{p_2}\end{cases}}{2\, r\, \omega_n - \sum_{i=1}^4 \,\omega_{p_i}}
\tag{2.144}
$$

Taking $p_4 = -(p_1 + p_2 + p_3)$, and using (2.141) we see that

$$
\delta E_1 \sim \sum_{p_1\, p_2\, p_3} \frac{1}{(p_1+p_2+p_3)^2}\frac{1}{p_1^2}\begin{cases}\frac{1}{p_2^{3/2}} \\ \frac{1}{p_2}\end{cases}
\tag{2.145}
$$

where all p_i are considered absolute valued, or equivalently the sum considered over positive integers. This is manifestly convergent. Continuing on to evaluate the leading x dependence, for the top choice in (2.144) we have poles for all three summation variables, while in the large x limit the K's go as constants, $G \sim 1/\sqrt{x}$ and the energy denominator is linear in

[14] Note that any contraction which would yield a delta function on the external state's spacetime indices is naturally zero here because we have chosen to analyze the traceless symmetric $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ state. It is a simple matter to analyze the trace state of section 2.3.1 here, and one finds convergence as well, however the number of (inverse) powers of summation variables will be 4 in the worst case, and thus the convergence is marginal. In no case does $\sqrt{\lambda'}$ behaviour occur here.

x, thus giving $\delta E_1 \sim 1/x$. For the bottom choice in (2.144), p_1 and p_3 have poles, while the sum over p_2 must be executed using (2.105). The scaling turns out identical however. Thus $\delta E_1/\mu$ is convergent and $\mathcal{O}(\lambda')$. One can repeat this argumentation for the second intermediate state in (2.142) and find the same behaviour. Also the entire exercise may be repeated for δE_2 in (2.139) using the following intermediate states

$$\begin{aligned} |I\rangle &\sim \alpha^\dagger_{p_1}\alpha^\dagger_{p_2}\alpha^\dagger_{p_3}\alpha^\dagger_{p_4}|\alpha_1\rangle|\alpha_2\rangle \\ |I\rangle &\sim \alpha^\dagger_{p_1}\alpha^\dagger_{p_2}\beta^\dagger_{p_3}\beta^\dagger_{p_4}|\alpha_1\rangle|\alpha_2\rangle \end{aligned} \tag{2.146}$$

and one discovers the same behaviour. The essential point is that we will always have at least 5 (inverse) powers of the summation variables, while the number of summation variables is 3. Alternate positionings of the oscillators in the intermediate states such as $|I\rangle \sim \alpha^\dagger_{p_1}\alpha^\dagger_{p_2}|\alpha_1\rangle\,\alpha^\dagger_{p_3}\alpha^\dagger_{p_4}|\alpha_2\rangle$ only improves the convergence, since level matching removes one more summation variable in these cases.

We can now consider adding additional pairs of fermionic and bosonic impurities to the intermediate state $|I\rangle$. This will add two factors of $\widetilde{N}^{1\,1}_{p_i\,p_j}$ or two factors of $G^{(1)}_{p_i}G^{(1)}_{p_j}$ (or equivalently two factors of $\widehat{Q}^{1\,1}_{p_ip_j}$). Either way the number of powers of summation variables increases in concert with the number of summation variables, preserving the convergence. Similarly the leading behaviour in λ' is unaffected. So it would seem that there are $\mathcal{O}(\lambda')$ contributions to $\delta E^{(2)}/\mu$ at every order in impurities, however any non-perturbative $\sqrt{\lambda'}$ behaviour is absent.

2.3.4 Summary and conclusions

We have presented an important set of results regarding the impurity non-conserving channel in light-cone string field theory on the plane-wave. The original expectation [81] that the four-impurity channel would lead as $\sqrt{\lambda'}$ has been contradicted; we find that this behaviour (for any number of intermediate impurities) is a manifestation of log-divergent mode number sums present in equal and opposite amounts in the standard H_3 and contact terms of the string field theory. This result is pleasing for two reasons: 1) because string amplitudes must be finite and 2) because $\sqrt{\lambda'}$ behaviour would present a serious challenge for reproduction in the gauge theory. A further result of our analysis is that, generically, all intermediate states contribute to the leading λ' term in the mass shift. This result is disturbing because the prospects of calculating the full shift for all channels is at least daunting, if not impossible. On the other hand it may explain the discrepancy between the impurity conserving result and that from the gauge theory. Physically, it seems non-sensical that an intermediate string with an arbitrarily high energy is equally as important as one whose energy is commensurate with the external state whose mass is receiving the correction. Indeed, at some point one would have to concern themselves with backreaction on the geometry. The analysis in section (2.3.3) is very generic; a cancellation mechanism could be hiding in the vertices which kill off powers of λ' as the number of intermediate-state impurities is raised. This is a very interesting direction to explore.

The reader may be concerned about details having been swept under the rug. We mentioned various intermediate states which we claimed, without demonstration, were non-divergent. One might also hope to find the coefficients of the leading λ' term to be numerically suppressed as the impurity non-conservation is increased. In section 2.5, an honest four-impurity calculation is presented where all contributions to the leading λ' result have been properly rendered. The result is non-divergent, non-zero, and not obviously suppressed numerically.

Finally, we also find that the reflection symmetry factor of $\frac{1}{2}$ dressing the H_3 term in [81], and argued incorrect in [86], must indeed be incorrect: dressing this term ruins the cancellation of log-divergences and renders the theory in-finite.

2.4 Calculation of the mass-shift via alternate vertices

This section is a presentation of the author's original work published in arXiv:hep-th/0605080 [90].

The construction of the light-cone string field theory given in section 2.2.3, i.e. (2.68), is not unique. The construction was guided by ensuring conservation of momentum, or that the string worldsheets touch at $\tau = 0$, which gave the exponential factors $|V\rangle$, and by the requirement that the supersymmetry algebra was obeyed, which determined the prefactors. The exponential factors are the unique method of ensuring (super)-locality while the prefactors in (2.68) are but one possible solution. The literature contains two others, one due to Di Vecchia, Petersen, Petrini, Russo, and Tanzini or DVPPRT [83] and another due to Dobashi and Yoneya or DY [57]. The vertex we have used thus far was developed by Spradlin, Volovich [53, 54], and by Stefanski and Pankiewicz [78, 84] and so we refer to it as SVPS. In [90], the author of this thesis calculated the impurity-conserving channel contribution to the mass shift stemming from these three choices of vertex. It was found that all vertices respected the same divergence cancellation mechanism and are therefore finite and lead as λ'. The DY vertex produced the best agreement with gauge theory, correctly reproducing the leading $\mathcal{O}(g_2^2\lambda')$ term.

2.4.1 The DVPPRT vertex

Introduction

In the construction of the SVPS vertex (2.68), an important point was glossed over. The symmetry group of the plane-wave background (2.1) is broken from $SO(8)$ to $SO(4)\times SO(4)\times \mathbb{Z}_2$ by the presence of the Ramond-Ramond field. The presence of such a field is well-known for complicating the quantization of worldsheet fermions; here it creates an ambiguity in the $\mathbb{Z}_2$-parity of the fermionic ground state. The trouble is found in the strange re-organizing of the fermionic modes given in (2.28), (2.29). In fact, the original treatment [60] followed a more usual procedure, defining creation and annihilation operators $\{\theta^a_n, \theta^{b\,\dagger}_m\} = \delta^{ab}\delta_{mn}$

$$\vartheta^a_n = \sqrt{\frac{\alpha'|n|}{2\,\omega_n|\alpha|}}\left(\theta^a_n + \theta^{a\,\dagger}_{-n}\right), \qquad \lambda^a_n = \sqrt{\frac{\omega_n|\alpha|}{2\,\alpha'|n|}}\left(\theta^a_{-n} + \theta^{a\,\dagger}_{n}\right) \tag{2.147}$$

and a "vacuum" state

$$\theta^a_n|\widetilde{0}\rangle = 0, \qquad a^I_n|\widetilde{0}\rangle = 0 \tag{2.148}$$

which is precisely what one would do in flat space. In the plane-wave background, this "vacuum" is not a zero-energy state, indeed

$$H_2\,|\widetilde{0}\rangle = 4\mu\,|\widetilde{0}\rangle. \tag{2.149}$$

The vacuum (2.34) is related to this vacuum via

$$|0;\alpha\rangle = \theta^5_0\,\theta^6_0\,\theta^7_0\,\theta^8_0|\widetilde{0}\rangle \tag{2.150}$$

i.e. the difference lies in the fermion zero modes. In [74], it was noted that the two vacuua have opposite $\mathbb{Z}_2$ parity for this reason. In attempting to preserve a smooth limit to flat space, the SVPS construction chooses

$$\mathbb{Z}_2\,|\widetilde{0}\rangle = |\widetilde{0}\rangle, \qquad \mathbb{Z}_2\,|0;\alpha\rangle = -|0;\alpha\rangle \tag{2.151}$$

and so must use $\mathbb{Z}_2$-odd prefactors in the interaction vertices (2.68). The DVPPRT vertex chooses the opposite, forsaking the smooth continuation to flat space as $\mu \to 0$, and requiring $\mathbb{Z}_2$-even prefactors. These they construct in the simplest possible way, by directly employing the quadratic Hamiltonian and supercharges

$$\begin{aligned}
|H_3^{\mathrm{DVPPRT}}\rangle &= \theta\,H_2|V\rangle = f\,\frac{g_2\,\alpha'}{16\,\alpha_3^3}\left(K^I K^I + \widetilde{K}^I\widetilde{K}^I + \text{fermions}\right)|V\rangle \\
|Q^{\mathrm{DVPPRT}}_{3\,\beta_1\dot\beta_2}\rangle &= \theta\,Q_{2\,\beta_1\dot\beta_2}|V\rangle = f\,\frac{g_2\,\eta}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\left(K^{\dot\gamma_1}_{\beta_1}\,Z_{\dot\gamma_1\,\dot\beta_2} - iK^{\gamma_2}_{\dot\beta_2}\,Y_{\beta_1\,\gamma_2}\right)|V\rangle
\end{aligned} \tag{2.152}$$

where $\theta = -g_2\,f\,r(1-r)/4$, and we have not explicitly calculated the fermionic portion of the H_3 prefactor as it will not concern us in the following calculations. It is obvious that these vertices obey the superalgebra; they have inherited that property from the free generators H_2 and Q_2.

Divergence cancellation

We would like to verify that the divergence cancellation mechanism found in section 2.3.1 for the SVPS vertex is also at play here. Unlike the SVPS case, the H_3 divergence does not stem from the two-bosonic-impurity intermediate state. There is, however, another divergence that was not present in the SVPS case. It is due to the contribution coming from matrix elements with two fermionic impurities in the intermediate state. In particular, the relevant matrix elements are given by

$$\begin{aligned}
&\langle\alpha_3|\alpha^i_n\alpha^i_{-n}\,\langle\alpha_2|\langle\alpha_1|\beta^{\alpha_1\alpha_2}_{p(1)}\beta_{-p(1)\beta_1\beta_2}|H_3^{\mathrm{DVPPRT}}\rangle = \\
&\qquad 4\,g_2 r\,(1-r)\left(\frac{\omega^{(3)}_n}{\alpha_3} + \frac{\omega^{(1)}_p}{\alpha_1}\right)\widehat{Q}^{1\,1}_{-p\,p}\widetilde{N}^{3\,3}_{-n\,n}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}
\end{aligned} \tag{2.153}$$

and similarly for the intermediate state with dotted indices. The divergent contribution to the energy shift coming from these matrix elements is found (by taking the large p limits of the summands) to be

$$\delta E^{\rm div}_{H_3} \sim -\frac{1}{2}\int_0^1 dr\ \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{3\,3}_{n\,-n}\right)^2 \sum_p \frac{1}{|p|}. \tag{2.154}$$

The contribution from the contact term stems from the following matrix element

$$\begin{aligned}\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1} \langle\alpha_3|\alpha^i_n\alpha^i_{-n}\,\langle\alpha_2|\langle\alpha_1|\alpha_p^{K\,(1)}\beta_{-p}^{(1)\,\Sigma_1\,\Sigma_2}|Q^{\rm DVPPRT}_{3\,\beta_1\dot\beta_2}\rangle =\\ 2\left(G^{(1)}_{|p|}\,K^{(3)}_{-n}\widetilde{N}^{3\,1}_{n\,p}+G^{(1)}_{|p|}\,K^{(3)}_{n}\widetilde{N}^{3\,1}_{-n\,p}\right)(\sigma^k)^{\dot\sigma_1}_{\beta_1}\delta^{\dot\sigma_2}_{\dot\beta_2}+8\,G^{(1)}_{|p|}\,K^{(1)}_p\widetilde{N}^{3\,3}_{n\,-n}(\sigma^K)^{\Sigma}_{\beta}\delta^{\Sigma}_{\beta}.\end{aligned} \tag{2.155}$$

The divergent contribution to the energy shift is found to be

$$\delta E^{\rm div}_{H_4} \sim +\int_0^1 dr\ \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{3\,3}_{n\,-n}\right)^2 \sum_{p>0} \frac{1}{p}. \tag{2.156}$$

Noting that in the H_3 contribution the divergence is found for both positive and negative p, while in the H_4 contribution the divergence occurs only for negative p, and hence a relative factor of 2 is induced in the H_3 term, one sees that the logarithmically divergent sums cancel identically between the H_3 and contact terms, leaving a convergent sum. This result can be generalized to arbitrary impurity channels, as was done for the SVPS case in section 2.3.3.

Impurity-conserving mass-shift

We now present the calculation of the impurity-conserving channel contribution to the mass shift of the $|[\mathbf{9},\mathbf{1}]\rangle$ state (2.94). The general method is outlined in detail in appendix D, section D.5. Beginning with the H_3 term of the mass-shift, we find[15]

$$\begin{aligned}\delta E^{\rm DVPPRT}_{H_3} = \frac{2}{r(1-r)}\frac{g_2^2\,\alpha'^2}{64\,\alpha_3^6}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\Bigg[\left(\widetilde{L}^{3\,r_1}_{n\,q_1}\right)^2\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2+\widetilde{L}^{3\,r_1}_{n\,q_1}\widetilde{L}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_2}_{-n\,q_2}\widetilde{N}^{3\,r_1}_{-n\,q_1}\\ +\widetilde{L}^{3\,r_1}_{-n\,q_1}\widetilde{L}^{3\,r_1}_{n\,q_1}\widetilde{N}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_2}_{-n\,q_2}+\widetilde{L}^{3\,r_1}_{-n\,q_1}\widetilde{L}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_1}_{-n\,q_1}\Bigg]\\ \times\frac{-\alpha_3\,(\delta^{r_1\,r_2}\delta_{q_1+q_2}+(1-\delta^{r_1\,r_2})\delta_{q_1}\delta_{q_2})}{2\,\omega_n-\beta^{-1}_{r_1}\omega_{q_1}-\beta^{-1}_{r_2}\omega_{q_2}}+(n\leftrightarrow -n)\end{aligned} \tag{2.157}$$

where the $L^{3\,r}_{n\,q}$'s are defined in (C.9). The sums are evaluated using (2.106); the result is[16]

[15] There is an implicit division of the energy-shift δE by the parameter μ in the remainder of the text. We have also dropped the integration $\int_0^1 dr$, which is implied in all subsequent amplitudes.

[16] A systematic code was developed to take input of the form (2.157) and to produce output of the form (2.158). The code was used for all calculations in this section. It correctly reproduces by-hand calculations and so we are confident it is accurate.

$$
\begin{aligned}
\delta E_{H_3}^{\mathrm{DVPPRT}} = \frac{g_2^2}{32\pi^2}\Bigg[& -\left(\frac{2}{3}+\frac{5}{4\pi^2 n^2}\right)\lambda' + 3\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} + n^2\left(1-\frac{9}{8\pi^2 n^2}\right)\lambda'^2 \\
& - 5n^2\left(\frac{2}{\pi^2}+\frac{3}{4\pi}\right)\lambda'^{5/2} - 5n^4\left(\frac{1}{4}-\frac{21}{32\pi^2 n^2}\right)\lambda'^3 \\
& + n^4\left(\frac{105}{16\pi}+\frac{94}{5\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg].
\end{aligned} \tag{2.158}
$$

Continuing with the contact term, we find

$$
\begin{aligned}
\delta E_{H_4}^{\mathrm{DVPPRT}} = -\frac{g_2^2\,\alpha'}{16\,\alpha_3^3}\sum_{r_1\,r_2}\sum_{q_1\,q_2} & \left[(K_n)^2\,(G_{q_1})^2\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2 + K_n K_{-n}\,(G_{q_1})^2\,\widetilde{N}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_2}_{-n\,q_2}\right] \\
& \times\left(\delta^{r_1\,r_2}\delta_{q_1+q_2} + (1-\delta^{r_1\,r_2})\delta_{q_1}\delta_{q_2}\right) + (n\leftrightarrow -n)
\end{aligned} \tag{2.159}
$$

with result

$$
\begin{aligned}
\delta E_{H_4}^{\mathrm{DVPPRT}} = \frac{g_2^2}{32\pi^2}\Bigg[& \left(\frac{1}{3}+\frac{5}{8\pi^2 n^2}\right)\lambda' - \frac{3}{2}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - n^2\left(\frac{1}{6}-\frac{19}{16\pi^2 n^2}\right)\lambda'^2 \\
& + n^2\left(\frac{11}{4\pi^2}+\frac{9}{8\pi}\right)\lambda'^{5/2} + \frac{n^4}{8}\left(1-\frac{105}{8\pi^2 n^2}\right)\lambda'^3 \\
& - n^4\left(\frac{45}{32\pi}+\frac{73}{20\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg].
\end{aligned} \tag{2.160}
$$

Adding the contributions from the H_3 and contact terms, we find the complete shift to be [90]

$$
\boxed{
\begin{aligned}
\delta E^{\mathrm{DVPPRT}} = \frac{g_2^2}{4\pi^2}\Bigg[& -\left(\frac{1}{24}+\frac{5}{64\pi^2 n^2}\right)\lambda' + \frac{3}{16}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} \\
& + n^2\left(\frac{5}{48}+\frac{1}{128\pi^2 n^2}\right)\lambda'^2 - n^2\left(\frac{29}{32\pi^2}+\frac{21}{64\pi}\right)\lambda'^{5/2} \\
& + n^4\left(-\frac{9}{64}+\frac{105}{512\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{303}{160\pi^2}+\frac{165}{256\pi}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg].
\end{aligned}
} \tag{2.161}
$$

This result does not fare very well in agreeing with the gauge theory result (2.15). It would seem that the DVPPRT vertex is either not correct or not complete based on this test of it. In the next section we will repeat the calculation using still another vertex. There we will find the best agreement with gauge theory yet found.

2.4.2 The "holographic" DY vertex

Introduction

The basis of the AdS/CFT correspondence is the GKP-W relation (1.102) discussed in section 1.5.5. In the plane-wave background, a set of supergravity states is singled-out from the full set of $AdS_5 \times S^5$ states. These plane-wave states have large angular momentum J on the five-sphere. From the perspective of the AdS_5 space, these are very massive Kaluza-Klein states, with $m^2 = J(J-4)$. The picture developed in section 1.5.5 was that the two point function of the dual CFT operators should be envisaged as a process whereby the insertion of the first operator on the boundary of AdS_5 causes the propagation of a supergravity mode into the bulk, which then turns back again and joins the second operator. For a sufficiently heavy supergravity mode, this propagation does not stray far from the classical geodesic joining the insertion points of the operators on the boundary. Technically, this means that the semi-classical action is dominated by a saddle which is the geodesic trajectory. Since in the plane-wave limit we are taking the mass of the supergravity modes to be very large, such a semi-classical treatment should be valid.

When we introduced the GKP-W relation in section 1.5.5, we chose to analyze the equations in Euclidean signature. In fact, we will see below that the picture of propagation from boundary to boundary makes little sense without requiring this signature. Recall that the metric of AdS_5 in global coordinates may be expressed as

$$ds^2 = \frac{1}{\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta\, d\Omega_3^2\right). \tag{2.162}$$

To solve the massive geodesics (for pure radial motion, i.e. $d\Omega_3^2 = 0$) is not terribly difficult. First we impose the constraint that the five-velocity squares to -1, while we also note that independence of t gives $\dot{t} = \xi \cos^2\theta$, then

$$-\cos^2\theta = -\xi^2\cos^4\theta + \dot{\theta}^2 \qquad \rightarrow \qquad \frac{\dot{\theta}}{\dot{t}} = \frac{d\theta}{dt} = \frac{\sqrt{\cos^2\theta - \xi^{-2}}}{\cos\theta}. \tag{2.163}$$

Integrating we find

$$\sin\theta = \sqrt{1-\xi^{-2}}\,\sin t, \qquad \xi > 1. \tag{2.164}$$

The boundary of AdS_5 sits at $\theta = \pi/2$, or $\sin\theta = 1$. Thus a massive geodesic never reaches the boundary; it turns back into the bulk at some $\theta < \pi/2$. Further, if we allowed for some angular motion, we would find that at its closest approach, the turning point, the particle's motion is *parallel* to the boundary. This seems to contradict the picture of particles originating from the boundary and propagating into the bulk.

Now consider the massive geodesic in the Euclidean picture; the negative signs in the equation on the left-hand side of (2.163) will be flipped to positive. The right-hand side becomes

$$\frac{\dot{\theta}}{\dot{t}} = \frac{d\theta}{dt} = \frac{\sqrt{-\cos^2\theta + \xi^{-2}}}{\cos\theta}, \qquad \xi < 1 \tag{2.165}$$

which has as solution

$$\sin\theta = \sqrt{\xi^{-2}-1}\cosh\left(t-t_0\right), \qquad t_0 = \ln\sqrt{\xi^{-2}-1}. \tag{2.166}$$

This geodesic reaches to the boundary and terminates normal to it; this is consistent with the GKP-W picture, see figure 2.8. For this reason, the AdS/CFT correspondence is usually

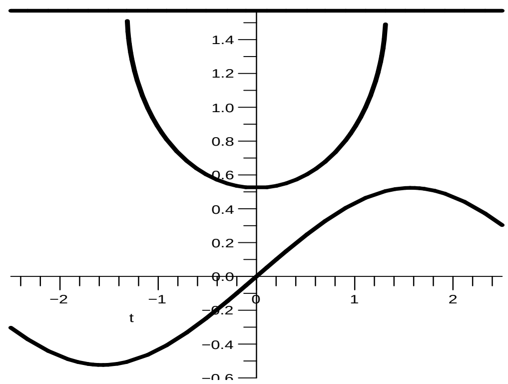


Figure 2.8: Massive geodesics in AdS are shown for Lorentzian (sinusoidal) and Euclidean (catenary) signatures. The axes are $\theta(t)$ vs. t, see (2.162). The boundary of AdS is the horizontal line at $\theta = \pi/2$. Only the Euclidean trajectory is consistent with the GKP-W picture.

stated as a relation between Euclidean CFT correlators and processes occurring in Euclidean AdS.

In [76], these Euclidean geodesics were interpreted as quantum mechanical tunnelling trajectories. Consider the Poincaré patch of Lorentzian AdS

$$ds^2 = R^2\frac{dz^2}{z^2} + \frac{1}{R^2z^2}\left(d\vec{x}^2 - dt^2\right). \tag{2.167}$$

The field equation for a scalar $\Phi(z,\vec{x}=0,t) = e^{i\omega t}\phi(z)$ of mass $m^2 = J(J-4)$ is

$$\left(z^2\partial_z^2 - 3\,z\,\partial_z + R^4z^2\omega^3 - J(J-4)\right)\phi(z) = 0. \tag{2.168}$$

Consider now applying a WKB approximation $\phi(z) = G(z)\exp(iS(z))$; the leading function $G(z)$ is of order 1, while the phase $S(z)$, being proportional to the potential is of order J. Note also that in the plane-wave limit $J \sim R^2$. Keeping the leading terms only, the result of plugging the WKB form into (2.168) is

$$z^2\left(\frac{dS}{dz}\right)^2 - R^4z^2\omega^2 + J^2 = 0, \qquad \rightarrow \qquad \frac{dS}{dz} = \sqrt{R^4\omega^2 - \frac{J^2}{z^2}} \tag{2.169}$$

and therefore $S(z)$ is real only if $z^2 \geq J^2/(\omega^2R^4)$; the boundary at $z=0$ is obviously excluded. However, the field is free to tunnel to the boundary, i.e. we may let $S(z)$ become imaginary. The tunnelling trajectory can then be found by noting

$$\phi^*(z)P_z\phi(z) = \phi^*(z)(-i\partial_z)\phi(z) = |\phi|^2\frac{dS}{dz} \tag{2.170}$$

and therefore associating the momentum along z to this via

$$\frac{dS}{dz} = \frac{R^2}{z^2}\frac{dz}{d\tau} \simeq \frac{J}{z^2}\frac{dz}{d\tau} \tag{2.171}$$

where τ is the affine parameter along the trajectory. This gives (for the tunnelling solution)

$$\frac{dz}{d\tau} = \pm z\sqrt{1-\frac{z^2\omega^2R^4}{J^2}} \tag{2.172}$$

which integrates to

$$z = \frac{J}{R^2\omega\cosh\tau} \tag{2.173}$$

reproducing the catenary shown in figure 2.8. Replacing $\tau \to i\tau$ in (2.173) reproduces the Lorentzian geodesic (2.164) or equivalently corresponds to the propagating solution for $\phi(z)$. However, note that since $J \sim d\psi/d\tau$, where ψ is an angle in S^5, and $\omega \sim dt/d\tau$, where t is the time coordinate of the boundary CFT, such a rotation would need to be accompanied by a double Wick rotation of the $AdS_5\times S^5$ metric in which both t and ψ become imaginary. This means that the tunnelling picture may be derived from the standard Lorentzian $AdS_5 \times S^5$ through double Wick rotation of ψ and t.

In the work [57], Dobashi and Yoneya continued this picture of holography to a construction of the light-cone string field theory vertices for the plane-wave background. They calculated the effective action for a massive scalar field along the aforementioned tunnelling trajectory. In their analysis, the first $SO(4)$ excitations come directly from harmonic oscillator ground states where the frequency of the harmonic oscillator is given by the mass of the supergravity state (i.e. $m^2 = \Delta(\Delta-4)$). The other $SO(4)$, associated with D_iZ insertions in the BMN operators, stem from excited states of these harmonic oscillators. The result is that the cubic coupling of the excited states is dictated completely by the cubic coupling of the ground states. This makes a definite prediction for the zero-mode sector of the string field theory; the cubic Hamiltonian H_3 ought to only count excitations of the first $SO(4)$. Of course this explicitly breaks the $\mathbb{Z}_2$ symmetry of the plane-wave background. The perspective is that this symmetry is “accidental” from the point of view of holography, and indications that it is not manifest at the level of CFT three-point functions were discovered already in [91]. We will endeavour to give a concise summary of the construction of the DY vertex. The effective action for computing the three-point functions of SUGRA scalars was worked-out in [32]. It is given by

$$S = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g}\left[\frac{1}{2}\left(\nabla\phi^i\right)^2 + \frac{1}{2}m_i^2\left(\phi^i\right)^2 - \frac{1}{3}G_{ijk}\phi^i\phi^j\phi^k\right] \tag{2.174}$$

and leads to agreement (via the GKP-W relation) with three-point functions of the dual CFT operators $\mathcal{O}_{\Delta_i}(x)$, where $m_i^2 = \Delta_i(\Delta_i-4)$,

$$\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{2\alpha_3}|x_2-x_3|^{2\alpha_1}|x_3-x_1|^{2\alpha_2}} \tag{2.175}$$

where $\alpha_1 = (\Delta_2 + \Delta_3 - \Delta_1)/2$, and similarly for α_2 and α_3. The coefficient C_{123} is given by a Δ_i-dependent constant multiplied by the cubic coupling G_{123}

$$C_{123} = \mathcal{N}(\Delta_1, \Delta_2, \Delta_3) G_{123}. \tag{2.176}$$

The strategy of Dobashi and Yoneya is to expand (2.174) about the tunnelling trajectory, quantize the free part of the action using creation/annihilation operators, and then to calculate the matrix elements of the cubic Hamiltonian stemming from (2.174). Let τ be the affine parameter along the tunnelling trajectory, while $\vec{y} = (\vec{x}, z)$ are the fluctuations in the given coordinates (see (2.167)). The effective metric is then [57]

$$ds^2 = (1 + \vec{y}^2) d\tilde{\tau}^2 + d\vec{y}^2, \qquad \tau = \tilde{\tau} + \frac{\vec{y}^2}{2} \tanh \tilde{\tau} \tag{2.177}$$

while the free part of the effective action becomes

$$\frac{4N^2}{(2\pi)^5} \int d\tilde{\tau} d^4 y \left[\left(1 - \frac{1}{2} y^2\right) \partial_{\tilde{\tau}} \bar{\Phi}_i \partial_{\tilde{\tau}} \Phi_i + \partial_y \bar{\Phi}_i \partial_y \Phi_i + \left(1 + \frac{1}{2} y^2\right) \Delta_i (\Delta_i - 4) \bar{\Phi}_i \Phi_i \right] \tag{2.178}$$

where $\Delta = J + k_i$ is the dimension of the BMN operator with k insertions of the $\mathcal{N} = 4$ SYM scalar $\Phi^i_{\mathcal{N}=4}$. Rewriting the fields as

$$\Phi_i = e^{-J\tilde{\tau}} \phi_0^{(J)}(\vec{y}) \psi(\tau), \qquad \bar{\Phi}_i = e^{J\tilde{\tau}} \phi_0^{(J)}(\vec{y}) \bar{\psi}(\tau) \tag{2.179}$$

where $\phi_0^{(J)}(\vec{y})$ is the ground state wave function of the operator $-\partial_y^2 + J^2 \vec{y}^2$

$$\phi_0^{(J)}(\vec{y}) = \left(\frac{J}{\pi}\right)^2 \exp\left(-\frac{1}{2} J \vec{y}^2\right) \tag{2.180}$$

allows the y-directions to be integrated-out, leaving the following free action for the $\psi(\tau)$ fields

$$\int d\tau \sum_i \left[\bar{\psi}_i \partial_\tau \psi_i - \partial_\tau \bar{\psi}_i \psi_i + k_i \bar{\psi}_i \psi_i \right] \tag{2.181}$$

and the following form for the interaction

$$\frac{1}{2} \int d\tau \sum_{i_1, i_2, i_3} \lambda_{i_1, i_2, i_3} \left(\bar{\psi}_{i_1} \psi_{i_2} \psi_{i_3} + \text{h.c.} \right) \tag{2.182}$$

where $\lambda_{i_1, i_2, i_3} = \mathcal{M}(\Delta_i) G_{i_1 i_2 i_3}$, where $\mathcal{M}(\Delta_i)$ is a constant dependent on the Δ_i, and we take $J_1 + J_2 = J_3$ to conserve angular momentum. By comparing λ_{123} with C_{123}, a direct map may be made between the $AdS_5 \times S^5$ couplings C_{123} and the (what ought to be) plane-wave cubic Hamiltonian coefficient λ_{123}. The result is that this coupling is proportional to $k_2 + k_3 - k_1$, i.e. the quadratic Hamiltonian counting the excitation energies of the BMN states

$$\lambda_{123} \propto k_2 + k_3 - k_1. \tag{2.183}$$

The other $SO(4)$'s worth of excitations, corresponding to insertions of $D_i Z$ in the BMN operators, are conjectured to correspond here to excited states of $\phi^{(J)}$

$$\phi_n^{(J)}(\vec{y}) = \prod_{i=1}^{4}\left(\frac{J}{\pi}\right)^{1/4}\frac{2^{-n_i/2}}{\sqrt{n_i!}}H_{n_i}(\sqrt{J}\vec{y})e^{-Jy^2/2}, \tag{2.184}$$

where the excitation number n_i corresponds to the insertion of n_i D_iZ's in the BMN operator. The crucial element here is that the couplings for the excited states are related directly to those of the ground states via

$$\lambda_{123}^{n_1 n_2 n_3} = \lambda_{123}^{000}\frac{\pi J_1}{J_2 J_3}\int d^4y\,\phi_{n_1}^{(J_1)}(\vec{y})\phi_{n_2}^{(J_2)}(\vec{y})\phi_{n_3}^{(J_3)}(\vec{y}). \tag{2.185}$$

The result is that the cubic Hamiltonian is still proportional only to the energies of the first $SO(4)$ excitations $k_2 + k_3 - k_1$.

At the end of the day, the vertex proposed by Dobashi and Yoneya must, at the level of supergravity states (i.e. string zero modes), count only the energies of the first $SO(4)$. This is accomplished by taking an average of the $\mathbb{Z}_2$-even prefactor of DVPPRT and the $\mathbb{Z}_2$-odd prefactor of SVPS. In this way the second $SO(4)$ zero modes cancel-out. The proposal is then

$$\begin{aligned} |H_3^{\mathrm{DY}}\rangle &= \frac{1}{2}\left(|H_3^{\mathrm{DVPPRT}}\rangle + |H_3^{\mathrm{SVPS}}\rangle\right) \\ |Q_3^{\mathrm{DY}}\rangle &= \frac{1}{2}\left(|Q_3^{\mathrm{DVPPRT}}\rangle + |Q_3^{\mathrm{SVPS}}\rangle\right) \end{aligned} \tag{2.186}$$

Divergence cancellation

The cancellation of divergences demonstrated in [89] for the SVPS vertex, was shown to extend to the DY vertex in [90]. In fact, we now show that an arbitrary linear combination of the SVPS and DVPPRT vertices,

$$H_3^N = \alpha\, H_3^{\mathrm{SVPS}} + \beta\, H_3^{\mathrm{DVPPRT}} \tag{2.187}$$

$$Q_3^N = \alpha\, Q_3^{\mathrm{SVPS}} + \beta\, Q_3^{\mathrm{DVPPRT}} \tag{2.188}$$

similarly yields a finite energy shift. We calculate the mass shift of the trace state as in section 2.3.1. The divergence stemming from the H_3 term is simply α^2 times the SVPS H_3 divergence (2.113) plus β^2 times the DVPPRT divergence (2.154). The reason is simple - the SVPS divergence stems from an entirely bosonic intermediate state, while (2.154) results from an entirely fermionic one. This precludes any divergences arising from cross-terms. We note that the SVPS divergence (2.116) is exactly equal to (2.154), therefore we have

$$\delta E_{H_3^N}^{\mathrm{div}} \sim -(\alpha^2+\beta^2)\frac{1}{2}\int_0^1 dr\,\frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{33}_{n\,-n}\right)^2\sum_p \frac{1}{|p|}. \tag{2.189}$$

The pieces of the SVPS Q_3 relevant to a two-impurity channel calculation are exactly Q_3^{DVPPRT} with $K \leftrightarrow \widetilde{K}$, therefore, from (2.115)

$$
\begin{aligned}
&\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1}\langle\alpha_3|\alpha_n^i\alpha_{-n}^i\,\langle\alpha_2|\langle\alpha_1|\alpha_p^{K\,(1)}\beta_{-p}^{(1)\,\Sigma_1\,\Sigma_2}|Q^{\mathrm{DVPPRT}}_{3\,\beta_1\dot\beta_2}\rangle=\\
&2G^{(1)}_{|p|}\left(\left[\alpha\,(K^{(3)}_{-n}\widetilde N^{3\,1}_{-n\,p}+K^{(3)}_{n}\widetilde N^{3\,1}_{n\,p})+\beta\,(K^{(3)}_{-n}\widetilde N^{3\,1}_{n\,p}+K^{(3)}_{n}\widetilde N^{3\,1}_{-n\,p})\right](\sigma^k)_{\beta_1}^{\dot\sigma_1}\delta^{\dot\sigma_2}_{\dot\beta_2}\right.\\
&\quad\left.+\,4\,(\beta\,K^{(1)}_{p}+\alpha\,K^{(1)}_{-p})\widetilde N^{3\,3}_{n\,-n}(\sigma^K)^{\Sigma}_{\beta}\delta^{\Sigma}_{\beta}\right)
\end{aligned}
\tag{2.190}
$$

The last term in (2.190) gives rise to a log-divergent sum, the large-p behaviour of which is

$$
\delta E^{\mathrm{div}}_{H_4^N}\sim+(\alpha^2+\beta^2)\int_0^1 dr\;\frac{g_2^2\,r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde N^{3\,3}_{n\,-n}\right)^2\sum_{p>0}\frac{1}{p}. \tag{2.191}
$$

Thus, by the same arguments as section 2.3.1, the energy shift is finite for arbitrary α and β. The DY vertex uses $\alpha=\beta=1/2$, and this combination exclusively gives rise to the agreement with gauge theory which will be presented in the next sub-section. Again, as for the DVPPRT vertex, the generalization of these arguments to the impurity non-conserving channels is a straightforward application of the treatment given in section 2.3.3.

Impurity-conserving mass-shift

In order to verify the validity of our results, we use two different methods for calculating the mass-shift. The first is straight-forward

$$
\delta E=\frac{\langle H_3^{\mathrm{DY}}|e\rangle\langle e|H_3^{\mathrm{DY}}\rangle}{E_0-H_2^{\mathrm{int}}}+\frac{1}{4}\langle Q_3^{\mathrm{DY}}|e\rangle\langle e|Q_3^{\mathrm{DY}}\rangle \tag{2.192}
$$

where $|e\rangle$ is the $|[\mathbf{9},\mathbf{1}]\rangle$ external state (2.94), and where the superscript "int" refers to internal states (i.e. strings number 1 and 2). For the second method, we recall that

$$
|H_3^{\mathrm{DY}}\rangle=\frac{1}{2}\,(\theta H_2\,|V\rangle+|H_3\rangle)\qquad |Q_3^{\mathrm{DY}}\rangle=\frac{1}{2}\,(\theta Q_2\,|V\rangle+|Q_3\rangle) \tag{2.193}
$$

where $\theta=-g_2r(1-r)/4$, and $|H_3\rangle$ and $|Q_3\rangle$ are the SVPS vertices (2.68). Because of the simple form of the DVPPRT vertices, a perhaps simpler form for the DY energy shift can be derived. We begin by considering some matrix elements

$$
\langle e|H_3^{\mathrm{DY}}\rangle=-\frac{\theta}{2}\,\Delta E\langle e|V\rangle+\frac{1}{2}\langle e|H_3\rangle \tag{2.194}
$$

$$
\langle e|Q_3^{\mathrm{DY}}\rangle=\frac{\theta}{2}\,\langle e|Q_2^{(3)}|V\rangle+\frac{\theta}{2}\,Q_2^{\mathrm{int}}\langle e|V\rangle+\frac{1}{2}\langle e|Q_3\rangle \tag{2.195}
$$

where $\Delta E=E_0-H_2^{\mathrm{int}}$, where E_0 is the energy of the external state. Plugging these into (2.192) we have, beginning with the H_3 term

$$
\delta E^{\mathrm{DY}}_{H_3}=\frac{\theta^2}{4}\,\Delta E\,\langle V|e\rangle\langle e|V\rangle+\frac{1}{4}\,\delta E^{\mathrm{SVPS}}_{H_3}-\frac{\theta}{4}\Big(\langle V|e\rangle\langle e|H_3\rangle+\mathrm{h.c.}\Big) \tag{2.196}
$$

and now for the contact term

$$\begin{aligned}\delta E_{H_4}^{\mathrm{DY}} &= \frac{\theta^2}{16}\left|\langle Q_2\, e|V\rangle\right|^2 + \frac{\theta^2}{16}\left|Q_2^{\mathrm{int}}\langle e|V\rangle\right|^2 + \frac{1}{4}\,\delta E_{H_4}^{\mathrm{SVPS}} \\ &+ \frac{\theta^2}{16}\Big(\langle V|Q_2\, e\rangle Q_2^{\mathrm{int}}\langle e|V\rangle + \mathrm{h.c.}\Big) + \frac{\theta}{16}\Big(\langle V|Q_2\, e\rangle\langle e|Q_3\rangle + \mathrm{h.c.}\Big) \\ &+ \frac{\theta}{16}\Big(\langle Q_3|e\rangle Q_2^{\mathrm{int}}\langle e|V\rangle + \mathrm{h.c.}\Big)\end{aligned} \tag{2.197}$$

where $|Q_2\, e\rangle = Q_2^{(3)}|e\rangle$. The second term of (2.197) can be combined with the first term of (2.196) by noting that

$$\frac{1}{4}Q_2^{\dagger\,\mathrm{int}}Q_2^{\mathrm{int}} = H_2^{\mathrm{int}} - \sum_{r=1}^{2}\frac{1}{\alpha_r}\sum_q q\, N_q^{(r)} \tag{2.198}$$

where $N_q^{(r)}$ is the number operator for string r and mode q. However, the level matching is true independently on each string, and so the extra term is zero. The result of adding the second term of (2.197) to the first term of (2.196) is thus:

$$\frac{\theta^2}{4}E_0\langle V|e\rangle\langle e|V\rangle \tag{2.199}$$

The last terms of (2.197) and (2.196) may also be combined. We note that,

$$\langle Q_3|e\rangle Q_2^{\mathrm{int}}\langle e|V\rangle = 4\langle H_3|e\rangle\langle e|V\rangle - \langle Q_3|Q_2\, e\rangle\langle e|V\rangle \tag{2.200}$$

The first term on the RHS will cancel the last term of (2.196). At the end of the day, the following expression for δE^{DY} may be used:

$$\begin{aligned}\delta E^{\mathrm{DY}} &= \frac{\theta^2}{4}E_0\langle V|e\rangle\langle e|V\rangle + \frac{\theta^2}{16}\left|\langle Q_2\, e|V\rangle\right|^2 + \frac{1}{4}\delta E^{\mathrm{SVPS}} \\ &+ \frac{\theta^2}{16}\Big(\langle V|Q_2\, e\rangle Q_2^{\mathrm{int}}\langle e|V\rangle + \mathrm{h.c.}\Big) + \frac{\theta}{16}\Big(\langle V|Q_2\, e\rangle\langle e|Q_3\rangle + \mathrm{h.c.}\Big) \\ &- \frac{\theta}{16}\Big(\langle V|e\rangle\langle Q_2\, e|Q_3\rangle + \mathrm{h.c.}\Big)\end{aligned} \tag{2.201}$$

which is the second method we have used to do the calculations. Both methods employ the general methodology of appendix D, section D.5.

First method

The H_3 contributions are of three varieties,

$$\delta E_{H_3} = \frac{1}{4}\delta E_{H_3}^{\mathrm{SVPS}} + \frac{1}{4}\delta E_{H_3}^{\mathrm{DVPPRT}} + \frac{1}{2}\frac{\langle H_3^{\mathrm{DVPPRT}}|e\rangle\langle e|H_3\rangle}{\Delta E}. \tag{2.202}$$

We find

$$
\begin{aligned}
\delta E_{H_3}^{\mathrm{SVPS}} = \frac{2}{r(1-r)} \frac{g_2^2\,\alpha'^2}{64\,\alpha_3^6} \sum_{r_1 r_2} \sum_{q_1 q_2} \Bigg[& \left(L_{n\,q_1}^{3\,r_1}\right)^2 \left(\widetilde{N}_{-n\,q_2}^{3\,r_2}\right)^2 + L_{n\,q_1}^{3\,r_1} L_{n\,q_2}^{3\,r_2} \widetilde{N}_{-n\,q_2}^{3\,r_2} \widetilde{N}_{-n\,q_1}^{3\,r_1} \\
& + L_{-n\,q_1}^{3\,r_1} L_{n\,q_1}^{3\,r_1} \widetilde{N}_{n\,q_2}^{3\,r_2} \widetilde{N}_{-n\,q_2}^{3\,r_2} + L_{-n\,q_1}^{3\,r_1} L_{n\,q_2}^{3\,r_2} \widetilde{N}_{n\,q_2}^{3\,r_2} \widetilde{N}_{-n\,q_1}^{3\,r_1} \Bigg] \\
& \times \frac{-\alpha_3\left(\delta^{r_1 r_2}\delta_{q_1+q_2} + (1-\delta^{r_1 r_2})\delta_{q_1}\delta_{q_2}\right)}{2\,\omega_n - \beta_{r_1}^{-1}\omega_{q_1} - \beta_{r_2}^{-1}\omega_{q_2}} + (n \leftrightarrow -n)
\end{aligned}
\tag{2.203}
$$

which evaluates to

$$
\begin{aligned}
\delta E_{H_3}^{\mathrm{SVPS}} = \frac{g_2^2}{32\pi^2} \Bigg[& \frac{15}{2\pi^2 n^2}\lambda' + 3\left(\frac{1}{\pi^2} + \frac{1}{2\pi}\right)\lambda'^{3/2} - \frac{27}{4\pi^2}\lambda'^2 - n^2\left(\frac{5}{\pi^2} + \frac{9}{4\pi}\right)\lambda'^{5/2} \\
& + \frac{111 n^2}{16\pi^2}\lambda'^3 + n^4\left(\frac{45}{16\pi} + \frac{33}{5\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4) \Bigg]
\end{aligned}
\tag{2.204}
$$

while for the DVPPRT vertex, we use the result (2.157), (2.158). Next we have the cross-term, the expression is

$$
\begin{aligned}
\delta E_{H_3}^{\mathrm{S\text{-}DV}} &= 2\frac{\langle H_3^{\mathrm{DVPPRT}}|e\rangle\langle e|H_3\rangle}{\Delta E} = \\
& \frac{2}{r(1-r)} \frac{g_2^2\,\alpha'^2}{32\,\alpha_3^6} \sum_{r_1 r_2} \sum_{q_1 q_2} \Bigg[\widetilde{L}_{n\,q_1}^{3\,r_1} L_{n\,q_1}^{3\,r_1} \left(\widetilde{N}_{-n\,q_2}^{3\,r_2}\right)^2 + \widetilde{L}_{n\,q_1}^{3\,r_1} L_{n\,q_2}^{3\,r_2} \widetilde{N}_{-n\,q_2}^{3\,r_2} \widetilde{N}_{-n\,q_1}^{3\,r_1} \\
& + \widetilde{L}_{-n\,q_1}^{3\,r_1} L_{n\,q_1}^{3\,r_1} \widetilde{N}_{n\,q_2}^{3\,r_2} \widetilde{N}_{-n\,q_2}^{3\,r_2} + \widetilde{L}_{-n\,q_1}^{3\,r_1} L_{n\,q_2}^{3\,r_2} \widetilde{N}_{n\,q_2}^{3\,r_2} \widetilde{N}_{-n\,q_1}^{3\,r_1} \Bigg] \\
& \times \frac{-\alpha_3\left(\delta^{r_1 r_2}\delta_{q_1+q_2} + (1-\delta^{r_1 r_2})\delta_{q_1}\delta_{q_2}\right)}{2\,\omega_n - \beta_{r_1}^{-1}\omega_{q_1} - \beta_{r_2}^{-1}\omega_{q_2}} + (n \leftrightarrow -n).
\end{aligned}
\tag{2.205}
$$

The result is

$$
\begin{aligned}
\delta E_{H_3}^{\mathrm{S\text{-}DV}} = \frac{g_2^2}{32\pi^2} \Bigg[& \left(\frac{8}{3} + \frac{20}{\pi^2 n^2}\right)\lambda' - 6\left(\frac{1}{\pi^2} + \frac{1}{2\pi}\right)\lambda'^{3/2} - n^2\left(\frac{8}{3} + \frac{14}{\pi^2 n^2}\right)\lambda'^2 \\
& + n^2\left(\frac{15}{\pi^2} + \frac{6}{\pi}\right)\lambda'^{5/2} + n^4\left(\frac{8}{3} + \frac{41}{4\pi^2 n^2}\right)\lambda'^3 \\
& - n^4\left(\frac{9}{\pi} + \frac{97}{4\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4) \Bigg].
\end{aligned}
\tag{2.206}
$$

Adding the contributions together we have

$$\delta E^{\text{DY}}_{H_3} = \frac{1}{4}\left(\delta E^{\text{SVPS}}_{H_3} + \delta E^{\text{DVPPRT}}_{H_3} + \delta E^{\text{S-DV}}_{H_3}\right) \tag{2.207}$$

and so the H_3 portion of the DY energy shift is given by

$$\begin{aligned}\delta E^{\text{DY}}_{H_3} = \frac{g_2^2}{4\pi^2}\Bigg[&\frac{3}{4}\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\lambda' - 5n^2\left(\frac{1}{96}+\frac{35}{256\pi^2 n^2}\right)\lambda'^2 \\ &+ n^4\left(\frac{17}{384}+\frac{655}{1024\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{3}{256\pi}+\frac{23}{640\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg].\end{aligned} \tag{2.208}$$

The contact term contributions are similarly of three varieties,

$$\delta E_{H_4} = \frac{1}{4}\delta E^{\text{SVPS}}_{H_4} + \frac{1}{4}\delta E^{\text{DVPPRT}}_{Q_3} + \frac{1}{8}\langle Q_3^{\text{DVPPRT}}|e\rangle\langle e|Q_3\rangle. \tag{2.209}$$

We find that

$$\begin{aligned}\delta E^{\text{SVPS}}_{H_4} = -\frac{g_2^2\,\alpha'}{16\,\alpha_3^3}\sum_{r_1 r_2}\sum_{q_1 q_2}&\left[(K_{-n})^2\,(G_{q_1})^2\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2 + K_{-n}K_n\,(G_{q_1})^2\,\widetilde{N}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_2}_{-n\,q_2}\right] \\ &\times\left(\delta^{r_1 r_2}\delta_{q_1+q_2} + (1-\delta^{r_1 r_2})\delta_{q_1}\delta_{q_2}\right) + (n\leftrightarrow -n)\end{aligned} \tag{2.210}$$

with result

$$\begin{aligned}\delta E^{\text{SVPS}}_{H_4} = \frac{g_2^2}{32\pi^2}\Bigg[&\left(\frac{1}{3}+\frac{5}{8\pi^2 n^2}\right)\lambda' - \frac{3}{2}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - n^2\left(\frac{1}{6}-\frac{19}{16\pi^2 n^2}\right)\lambda'^2 \\ &+ n^2\left(\frac{11}{4\pi^2}+\frac{9}{8\pi}\right)\lambda'^{5/2} + \frac{n^4}{8}\left(1-\frac{105}{8\pi^2 n^2}\right)\lambda'^3 \\ &- n^4\left(\frac{45}{32\pi}+\frac{73}{20\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg].\end{aligned} \tag{2.211}$$

Again, for the DVPPRT contributions we refer to (2.159) and (2.160). The expression for the cross-term is

$$\begin{aligned}\delta E^{\text{S-DV}}_{H_4} &= \frac{1}{2}\langle Q_3^D|e\rangle\langle e|Q_3\rangle = \\ -\frac{g_2^2\,\alpha'}{8\,\alpha_3^3}\sum_{r_1 r_2}\sum_{q_1 q_2}&\left[K_n K_{-n}\,(G_{q_1})^2\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2 + K_n K_n\,(G_{q_1})^2\,\widetilde{N}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_2}_{-n\,q_2}\right] \\ &\times\left(\delta^{r_1 r_2}\delta_{q_1+q_2} + (1-\delta^{r_1 r_2})\delta_{q_1}\delta_{q_2}\right) + (n\leftrightarrow -n)\end{aligned} \tag{2.212}$$

with result

$$
\begin{aligned}
\delta E_{H_4}^{\text{S-DV}} = \frac{g_2^2}{32\pi^2}\Bigg[& -2\left(\frac{1}{3}+\frac{5}{8\pi^2 n^2}\right)\lambda' + 3\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} + 4n^2\left(\frac{1}{6}+\frac{1}{2\pi^2 n^2}\right)\lambda'^2 \\
& - 2n^2\left(\frac{11}{4\pi^2}+\frac{9}{8\pi}\right)\lambda'^{5/2} - 4n^4\left(\frac{1}{6}+\frac{5}{16\pi^2 n^2}\right)\lambda'^3 \\
& + n^4\left(\frac{3}{\pi}+\frac{317}{40\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg].
\end{aligned}
\tag{2.213}
$$

Adding the contributions together we have

$$
\delta E_{H_4}^{\text{DY}} = \frac{1}{4}\left(\delta E_{H_4}^{\text{SVPS}} + \delta E_{Q_3}^{\text{DVPPRT}} + \delta E_{H_4}^{\text{S-DV}}\right) \tag{2.214}
$$

and so the H_4 portion of the DY energy shift is given by

$$
\begin{aligned}
\delta E_{H_4}^{\text{DY}} = \frac{g_2^2}{4\pi^2}\Bigg[& n^2\left(\frac{1}{96}+\frac{35}{256\pi^2 n^2}\right)\lambda'^2 - \frac{5n^4}{128}\left(\frac{1}{3}+\frac{29}{8\pi^2 n^2}\right)\lambda'^3 \\
& + \frac{n^4}{256}\left(\frac{3}{2\pi}+\frac{5}{\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg].
\end{aligned}
\tag{2.215}
$$

Assembling the H_3 and contact term results, we arrive at the final expression for the Yoneya energy shift

$$
\boxed{
\begin{aligned}
\delta E^{\text{DY}} = \frac{g_2^2}{4\pi^2}\Bigg[& \left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\frac{3}{4}\lambda' - \frac{n^2}{2}\lambda'^2\right) + \frac{n^4}{32}\left(1+\frac{255}{16\pi^2 n^2}\right)\lambda'^3 \\
& + \frac{n^4}{512}\left(\frac{9}{\pi}+\frac{142}{5\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg]
\end{aligned}
}
\tag{2.216}
$$

Second method

Referring to (2.201) we have five contributions to consider, beyond the SVPS result. In this section we enumerate these results and show that the final answers are in agreement with the first method calculations. The terms of (2.201) which are independent of the SVPS vertices lead individually like a constant, however together they lead as λ'. We therefore present the results for the sum of these terms. The remaining terms are individually of $\mathcal{O}(\lambda')$ and are presented individually. The terms independent of the SVPS vertices are

$$
\delta E_1 = \frac{\theta^2}{16}E_0\langle V|e\rangle\langle e|V\rangle + \frac{\theta^2}{16}\left|\langle Q_2\, e|V\rangle\right|^2 + \frac{\theta^2}{16}\Big(\langle V|Q_2\, e\rangle Q_2^{\text{int}}\langle e|V\rangle + \text{h.c.}\Big) \tag{2.217}
$$

where some of the relevant matrix elements can be found in appendix D. The resulting expressions are (for clarity we suppress the level-matching factor of $(\delta^{r_1 r_2}\delta_{q_1+q_2} + (1-\delta^{r_1 r_2})\delta_{q_1}\delta_{q_2})$ in the remainder of this section)

$$\frac{g_2^2\, r(1-r)}{32\,(-\alpha_3)}\,(\omega_n - n)\sum_{r_1\,r_2}\sum_{q_1\,q_2}\left[\left(\widetilde{N}^{3\,r_1}_{n\,q_1}\right)^2\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2 + \widetilde{N}^{3\,r_1}_{n\,q_1}\widetilde{N}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_2}_{-n\,q_2}\widetilde{N}^{3\,r_1}_{-n\,q_1}\right] + (n \leftrightarrow -n) \tag{2.218}$$

$$\frac{g_2^2\, r(1-r)}{64\,(-\alpha_3)}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\left[\left|\Omega_n\,\widehat{Q}^{3\,r_1}_{n\,q_1}\right|^2\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2 + \Omega_n\Omega_{-n}\widehat{Q}^{3\,r_1\,*}_{n\,q_1}\widehat{Q}^{3\,r_1}_{-n\,q_1}\widetilde{N}^{3\,r_2}_{n\,q_2}\widetilde{N}^{3\,r_2}_{-n\,q_2}\right] + (n \leftrightarrow -n) \tag{2.219}$$

$$\begin{aligned} -i\frac{g_2^2\, r(1-r)}{32\,(-\alpha_3)}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\Bigg[&\Omega_n\,\widehat{Q}^{3\,r_1\,*}_{n\,q_1}\frac{\Omega_{q_1}}{\sqrt{\beta_{r_1}}}\widetilde{N}^{3\,r_1}_{n\,q_1}\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2 \\ &+ \Omega_n\,\widehat{Q}^{3\,r_1\,*}_{n\,q_1}\frac{\Omega_{q_1}}{\sqrt{\beta_{r_1}}}\widetilde{N}^{3\,r_1}_{n\,q_1}\widetilde{N}^{3\,r_2}_{-n\,q_2}\widetilde{N}^{3\,r_2}_{n\,q_2}\Bigg] + (n \leftrightarrow -n) \end{aligned} \tag{2.220}$$

respectively. The result is

$$\begin{aligned} \delta E_1 = \frac{g_2^2}{32\pi^2}\Bigg[&-\left(\frac{1}{12}+\frac{5}{32\pi^2 n^2}\right)\lambda' + \frac{3}{8}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} + n^2\left(\frac{5}{24}+\frac{1}{64\pi^2 n^2}\right)\lambda'^2 \\ &- \frac{n^2}{16}\left(\frac{29}{\pi^2}+\frac{21}{2\pi}\right)\lambda'^{5/2} + \frac{n^4}{32}\left(-9+\frac{105}{8\pi^2 n^2}\right)\lambda'^3 \\ &+ n^4\left(\frac{165}{128\pi}+\frac{303}{80\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg]. \end{aligned} \tag{2.221}$$

The first remaining term is given by

$$\delta E_2 = \frac{\theta}{16}\Big(\langle V|Q_2\, e\rangle\langle e|Q_3\rangle + \text{h.c.}\Big) \tag{2.222}$$

which gives the following expression

$$i\frac{g_2^2}{16\alpha_3^3}\sqrt{\frac{\alpha'\kappa}{4\alpha_3}}\left[\Omega_n K_{-n} G_{q_1}\widehat{Q}^{3\,r_1\,*}_{n\,q_1}\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2 + \Omega_n K_n G_{q_1}\widehat{Q}^{3\,r_1\,*}_{n\,q_1}\widetilde{N}^{3\,r_2}_{-n\,q_2}\widetilde{N}^{3\,r_2}_{n\,q_2}\right] + (n \leftrightarrow -n) \tag{2.223}$$

yielding the result

$$\begin{aligned}\delta E_2 = \frac{g_2^2}{32\pi^2}\Bigg[&\left(\frac{1}{6}+\frac{35}{16\pi^2 n^2}\right)\lambda' - n^2\left(\frac{1}{6}+\frac{5}{4\pi^2 n^2}\right)\lambda'^2 \\ &+\frac{n^2}{2}\left(\frac{1}{\pi^2}+\frac{3}{8\pi}\right)\lambda'^{5/2}+n^4\left(\frac{1}{6}+\frac{31}{32\pi^2 n^2}\right)\lambda'^3 \\ &-n^4\left(\frac{3}{8\pi}+\frac{21}{20\pi^2}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\Bigg].\end{aligned} \tag{2.224}$$

The next term is

$$\delta E_3 = -\frac{\theta}{16}\Big(\langle V|e\rangle\langle Q_2\, e|Q_3\rangle + \text{h.c.}\Big) \tag{2.225}$$

which gives the expression

$$\frac{g_2^2\,\alpha'}{32\alpha_3^3}\left[K_n K_{-q_1}\widetilde{N}^{3\,r_1}_{n\,q_1}\left(\widetilde{N}^{3\,r_2}_{-n\,q_2}\right)^2 + K_n K_{-q_2}\widetilde{N}^{3\,r_2}_{-n\,q_2}\widetilde{N}^{3\,r_1}_{n\,q_1}\widetilde{N}^{3\,r_1}_{-n\,q_1}\right] + (n\leftrightarrow -n) \tag{2.226}$$

and results in

$$\begin{aligned}\delta E_3 = \frac{g_2^2}{32\pi^2}\Bigg[&\left(\frac{1}{3}+\frac{5}{2\pi^2 n^2}\right)\lambda' - \frac{3}{4}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - n^2\left(\frac{1}{3}+\frac{7}{4\pi^2 n^2}\right)\lambda'^2 \\ &+\frac{n^2}{4}\left(\frac{15}{2\pi^2}+\frac{3}{\pi}\right)\lambda'^{5/2}+n^4\left(\frac{1}{3}+\frac{41}{32\pi^2 n^2}\right)\lambda'^3 \\ &-n^4\left(\frac{9}{8\pi}+\frac{97}{32\pi^2}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\Bigg].\end{aligned} \tag{2.227}$$

Adding the contributions,

$$\delta E^{\text{DY}} = \frac{1}{4}\delta E^{\text{SVPS}} + \delta E_1 + \delta E_2 + \delta E_3 \tag{2.228}$$

we find the identical result (2.216).

2.4.3 Discussion

The expression for the DY vertex calculation of the impurity-conserving mass-shift (2.216), represents the best matching with the gauge theory result (2.15) yet achieved. There is a leading factor of $3/4$ in the λ' term which we can scale away by employing the undetermined function f which appeared in the vertices. If we scale f by $\sqrt{4/3}$ we will achieve agreement of the leading λ' term with gauge theory. Although the λ'^2 term is of the correct form, the coefficient is not in agreement. We also note the absence of half-integer powers of λ' up to (but not including) the 7/2's power. This fairs much better than the SVPS, and one half-power better than the DVPPRT results.

We showed in section 2.3 that any "reflection symmetry factor" which would effectively multiply the contact term by 2 relative to the H_3 term is incommensurate with finiteness of the mass-shift. Mysteriously, however, if the contact terms are blindly scaled by a factor of 2, the agreement with gauge theory is enhanced for both the SVPS (2.107) and DY results,

$$\begin{aligned}\delta E_{2H_4}^{\mathrm{SVPS}} = \frac{g_2^2}{4\pi^2}\Bigg[&\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\lambda' - \frac{n^2}{2}\lambda'^2\right) + \frac{n^2}{16\pi^2}\lambda'^{5/2} \\ &+ n^4\left(\frac{1}{32}+\frac{117}{256\pi^2 n^2}\right)\lambda'^3 - \frac{7n^4}{80\pi^2}\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg]\end{aligned} \tag{2.229}$$

$$\begin{aligned}\delta E_{2H_4}^{\mathrm{DY}} = \frac{g_2^2}{4\pi^2}\frac{3}{4}\Bigg[&\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\lambda' - \frac{n^2}{2}\lambda'^2\right) + n^4\left(\frac{7}{288}+\frac{365}{768\pi^2 n^2}\right)\lambda'^3 \\ &+ n^4\left(\frac{1}{10\pi^2}+\frac{1}{32\pi}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg]\end{aligned} \tag{2.230}$$

however, the DY result is still superior in that the $\lambda'^{5/2}$ power is absent. The meaning (if any) of this coincidence is not clear to us at this stage.

The DY vertex has thus produced the best match to gauge theory so far. It matches the λ' term, exhibits the correct form of the λ'^2 term, and displays the absence of half-integer powers of λ' to a rather high order. It is possible that higher-orders in intermediate state impurities would correct the result to a complete match with gauge theory; indeed a scheme whereby higher orders in impurities somehow contribute only higher orders in λ' would be very physical and pleasing. Whether or not this is the case remains to be seen and requires an honest calculation from these channels; as we have shown in section 2.3.3, generically this is not the case.

2.5 Wrapping x^-: discrete light-cone quantization

In an important paper by Mukhi, Rangamani, and Verlinde [106], a version of the plane-wave / BMN operator correspondence was derived whereby the light-cone direction x^- is compactified leading to a discrete light-cone momentum p^+. The dual gauge theory is no longer $\mathcal{N}=4$ supersymmetric Yang-Mills, but an $\mathcal{N}=2$ quiver gauge theory corresponding to a stack of N_1 D-branes at a $\mathbb{C}^3/\mathbb{Z}_{N_2}$ orbifold point. The plane-wave is obtained via a Penrose limit on $AdS_5 \times S_5/\mathbb{Z}_{N_2}$, where the five-sphere is orbifolded into N_2 domains.

The authors of [107] computed the non-planar corrections to the anomalous dimensions of the gauge theory operators corresponding to strings on the discrete light-cone plane-wave. It is therefore interesting to consider light-cone string field theory in this discrete light-cone quantization. This section presents original, unpublished work of the author and collaborators of [89] concerning this DLCQ light-cone string field theory.

2.5.1 Introduction

The space $AdS_5 \times S_5/\mathbb{Z}_{N_2}$ may be expressed as

$$ds^2 = R^2\Big[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + d\alpha^2 + \sin^2\alpha\, d\theta^2 + \cos^2\alpha\big(d\gamma^2 + \cos^2\gamma\, d\chi^2 + \sin^2\gamma\, d\phi^2\big)\Big] \quad (2.231)$$

where θ, χ, and ϕ are azimuthal angles $\in [0, 2\pi]$. The orbifold is realized by imposing the following identifications

$$\chi \sim \chi + \frac{2\pi}{N_2}, \qquad \phi \sim \phi - \frac{2\pi}{N_2}. \quad (2.232)$$

The Penrose limit is realized via the re-scalings $r = \rho R$, $w = \alpha R$, $y = \gamma R$, and the introduction of light-cone coordinates

$$x^+ = \frac{1}{2}(t + \chi), \qquad x^- = \frac{R^2}{2}(t - \chi). \quad (2.233)$$

Taking the limit $R \to \infty$, the plane-wave metric (2.1) is obtained, albeit with compactifications

$$x^+ \sim x^+ + \frac{\pi}{N_2}, \qquad x^- \sim x^- + \frac{\pi R^2}{N_2}. \quad (2.234)$$

If we then take $N_2 \to \infty$, the periodicity in x^+ and ϕ is removed, while scaling $N_2 \sim R^2$ causes a finite compactification of x^-. The implication for string theory is very simple; it is unchanged up to two important features

1. The light-cone momentum is quantized in units of inverse compactification radius $R_- = R^2/(2N_2)$

$$2p^+ = \frac{k}{R_-}, \qquad k \in \mathbb{Z},\ k > 0 \quad (2.235)$$

2. The string is free to wrap x^-, leading to a modified level matching

$$\prod_i a^{I\,\dagger}_{n_i^b} \prod_j b^{a\,\dagger}_{n_j^f} |0; k, m\rangle \to \sum_i n_i^b + \sum_j n_j^f = km, \qquad m \in \mathbb{Z} \quad (2.236)$$

where m is the wrapping number.

The dual gauge theory is constructed by considering N_1 coincident D3-branes sitting at a $\mathbb{C}^3/\mathbb{Z}_{N_2}$ orbifold point. There are thus N_2 copies of the N_1 branes. The gauge group of the un-orbifolded theory is then broken as follows

$$SU(N_1 N_2) \to SU(N_1)_1 \times SU(N_1)_2 \times \ldots \times SU(N_1)_{N_2} \quad (2.237)$$

so that there are now N_2 separate $SU(N_1)$ gauge groups. The action of the orbifold group generator Γ on the six scalars of the parent $\mathcal{N}=4$ SYM is as follows

$$\begin{aligned}\Gamma : &\left(\frac{1}{\sqrt{2}}(\Phi_1+i\Phi_2), \frac{1}{\sqrt{2}}(\Phi_3+i\Phi_4), \frac{1}{\sqrt{2}}(\Phi_5+i\Phi_6)\right)\\ &= \left(\frac{\omega}{\sqrt{2}}(\Phi_1+i\Phi_2), \frac{\omega^{-1}}{\sqrt{2}}(\Phi_3+i\Phi_4), \frac{1}{\sqrt{2}}(\Phi_5+i\Phi_6)\right)\end{aligned} \tag{2.238}$$

where $\omega = \exp(2\pi i/N_2)$. This leads to new bi-fundamental fields A_I, B_I which each have one leg each in $SU(N_1)_I$ and $SU(N_1)_{I+1}$, corresponding to the first and second combinations of the parent scalars, and complex scalars Φ_I in the adjoint representation of $SU(N_1)_I$, corresponding to the remaining parent scalar combination $\frac{1}{\sqrt{2}}(\Phi_5+i\Phi_6)$. The resulting gauge theory is known as a *quiver* theory (see [108]) and in this case carries half the supersymmetry of the parent theory. The relation (1.65) is not modified, i.e. we simply replace $N \longrightarrow N_1N_2$ so that

$$R^2 = \sqrt{4\pi g_s\,\alpha'^2 N_1N_2}, \qquad g_{YM}^2 = 4\pi g_s. \tag{2.239}$$

Given the scaling $N_2 \sim R^2$, we are instructed to take $N_1 \sim N_2$ so that g_s remains fixed. Each gauge group has a coupling constant given by $(g_{YM}^I)^2 = 4\pi g_s N_2$, so that the relevant 't Hooft coupling is $\lambda = (g_{YM}^I)^2 N_1 = 4\pi g_s N_1 N_2$.

Following the treatment of BMN given in section 2.1, we would like to identify the appropriate "large-J" limit of the orbifolded theory in order to identify the operators dual to DLCQ plane-wave strings. Two angular momenta are identified [106]

$$J = -\frac{i}{2N_2}\left(\partial_\chi - \partial_\phi\right), \qquad J' = -\frac{i}{2}\left(\partial_\chi + \partial_\phi\right) \tag{2.240}$$

so that the light-cone momenta are expressed as

$$\begin{aligned}2p^- &= i(\partial_t+\partial_\chi) = \Delta - N_2J - J'\\ 2p^+ &= i\frac{(\partial_t-\partial_\chi)}{R^2} = \frac{\Delta + N_2J + J'}{R^2}.\end{aligned} \tag{2.241}$$

In analogy with the BMN case, we would like to take Δ and N_2J+J' to infinity as R^2, while keeping their difference finite. The charges of the A_I, B_I, and Φ_I fields are as follows [106]

	Δ	N_2J	J'
A_I	1	1/2	1/2
B_I	1	−1/2	1/2
Φ_I	1	0	0

which indicates that the desired operators are long chains of A_I's (which have $\Delta = N_2J+J'$), with insertions of $\Phi_I, \bar{\Phi}_I, B_I, \bar{B}_I$ as the fundamental impurities which have $\Delta = 1$ while having $N_2J+J'=0$. The other $SO(4)$ impurities are constructed via insertions of derivatives of the A_I. In order that the operator be gauge invariant, the product must be over all N_2

copies of $SU(N_1)$. The simplest state is the dual of the DLCQ string vacuum. It is therefore not surprising to find

$$|k=1, m=0\rangle \leftrightarrow \frac{1}{\sqrt{N_1^{N_2}}}\,\mathrm{Tr}\,(A_1 A_2 \dots A_{N_2}) \tag{2.242}$$

where we note that the string vacuum must have $m=0$ (a string must exist in order to wrap a direction). For general k, the operator is

$$|k, m=0\rangle \leftrightarrow \frac{1}{\sqrt{N_1^{kN_2}}}\,\mathrm{Tr}\Big((A_1 A_2 \dots A_{N_2})^k\Big) \tag{2.243}$$

so that k copies of the string $A_1 \dots A_{N_2}$ are traced over. Adding impurities we see a novel feature as compared to the standard BMN picture. Consider the addition of a single impurity to the operator (2.242)

$$\left(a_n^{5\,\dagger} + i a_n^{6\,\dagger}\right)|k=1, m\rangle \leftrightarrow \sum_{I=1}^{N_2} e^{2\pi i n I/N_2}\,\mathrm{Tr}\,(A_1 \dots A_{I-1}\Phi_I A_I \dots A_{N_2})\,. \tag{2.244}$$

This would have been zero by cyclicity of the trace, but here each insertion position is inequivalent to the next, so that this state is non-zero. This is a wrapping state with $m=n$. For general k, we have

$$\left(a_n^{5\,\dagger} + i a_n^{6\,\dagger}\right)|k, m\rangle \leftrightarrow \sum_{I=1}^{kN_2} e^{2\pi i n I/(kN_2)}\,\mathrm{Tr}\left(A_1 \dots A_{I-1}\Phi_I A_I \dots A_{N_2}\,(A_1 \dots A_{N_2})^{k-1}\right) \tag{2.245}$$

where, since cyclicity gives the same trace under $I \to I + N_2$, n must be k times an integer. This is just the level matching condition $n = km$. The construction of higher impurity states is straightforward [106].

In [107], the DLCQ analogue of (2.15) was computed for one and two-impurity operators built upon $k = 1, 2$, and 3 vacuua. The couplings λ' and g_2 may be expressed in terms of N_1, N_2, and k using (2.235)

$$\begin{aligned} \alpha' p^+ &= \frac{\alpha' k}{2R_-} = \frac{\alpha' k N_2}{R^2} = \frac{k}{g_{YM}}\sqrt{\frac{N_2}{N_1}} \quad \to \quad \lambda' = \frac{1}{(\alpha' p^+)^2} = \frac{g_{YM}^2 N_1}{k^2 N_2} \\ g_2 &= g_{YM}^2(\alpha' p^+)^2 = \frac{k^2 N_2}{N_1} \end{aligned} \tag{2.246}$$

where we note that μ has been scaled out of the metric here. The results of [107] may be summarized as follows.

1. Single impurity operators receive only planar corrections to their anomalous dimension; this implies the absence of string-loop corrections to the masses of the dual string states. The planar loop corrections reproduce the expansion of the free string energy.

2. Two-impurity operators with $k = 1$ similarly receive no non-planar corrections. The planar loop corrections also reproduce the expansion of the free string energy.

3. Two-impurity operators with $k = 2$ receive the correct free-string planar corrections, but also receive a leading non-planar correction to the anomalous dimension

$$\Delta - N_2 J - J' = \left(2 + \frac{1}{2}(n_1^2 + n_2^2)\lambda' + \dots\right) + \begin{cases} g_2^2\left(\frac{1}{16\pi^2}\lambda' + \dots\right) & n_1, n_2 \text{ odd} \\ 0 & n_1, n_2 \text{ even} \end{cases} \tag{2.247}$$

which truncates at $\mathcal{O}(g_2^2)$. Note that n_1 and n_2 are the mode numbers of the dual string oscillators obeying the level matching condition $n_1 + n_2 = 2m$, where $m \in \mathbb{Z}$.

4. Two-impurity operators with $k = 3$ receive the correct free-string planar corrections, and also receive non-planar corrections to arbitrary order in g_2^2. The leading result is given by

$$\begin{aligned} \Delta - N_2 J - J' = &\left(2 + \frac{1}{2}(n_1^2 + n_2^2)\lambda' + \dots\right) \\ &+ \frac{g_2^2\lambda'}{16\pi^2}\left[1 + \frac{6}{\pi(n_1 - n_2)}\left(\cos\left(\frac{\pi n_1}{3}\right)\sin\left(\frac{\pi n_1}{3}\right) - \cos\left(\frac{\pi n_2}{3}\right)\sin\left(\frac{\pi n_2}{3}\right)\right)\right] \\ &+ \dots \end{aligned} \tag{2.248}$$

where **all** non-planar corrections (not just the leading term shown) vanish for n_1, n_2 multiples of three.

We also take $n_1 \neq n_2$ in all results shown here, i.e. $m \neq 0$. It is an interesting pursuit to attempt to calculate these non-planar corrections using string loops as has been attempted for the standard BMN operators in the previous sections. In the next section we will endeavour to reproduce (2.247), (2.248) and the results discussed under items 1) and 2) above using DLCQ light-cone string field theory on the plane-wave background.

2.5.2 Results

As we asserted in (2.235) and (2.236), the light-cone string field theory is unchanged in the DLCQ case, with the exception of a modified level-matching condition and a discretized p^+. Because p^+ is conserved and non-zero, the string with $k = 1$ cannot split, as there is no lower p^+ strings to split into; this is the dual-reflection of item 2) from the previous subsection. For the same reason, we see that once the $k = 2$ string is split into two $k = 1$ strings, the only choice is to re-join to a $k = 2$ state. Therefore, the mass-shift of a $k = 2$ string may not be of higher than g_2^2 order; as was summarized in item 3). The string theory manifestation of item 1) (that single-impurity states receive no loop corrections) is in fact also responsible for the lack of $k = 2$ corrections when both external mode numbers are even, or in the case of $k = 3$, when both are multiples of 3. The source is the factor of $\sin n\pi r$ which occurs in each Neumann matrix and associated quantity which has a leg in the external string (string #3), see appendix C. Recall that $r = \alpha_1/|\alpha_3| = k_1/k$, so that if $n_i k_1/k$ is an integer for some external string excitation $\alpha^\dagger_{n_i}$, the entire amplitude will vanish. This factor comes about from the decomposition of the modes of the external string into those of the two internal strings at $\tau = 0$, see figure 2.5. If the undulations of string #3 at $\tau = 0$ are orthogonal to those of strings #1 or #2, then obviously the string worldsheets cannot be in contact, and therefore cannot interact. This situation is realized if one of the n_i is a multiple of the external light-cone momentum k, i.e. for $k = 2$ when at least one n_i is even, for $k = 3$ when at least one n_i is a multiple of three, or, when only a single external impurity is present, always since $n = km$ by level-matching, where m is the external wrapping number.

$k = 2$ Impurity-conserving mass-shift

The calculation of the specific one-loop mass-shift for $k = 2$ proceeds along the same lines as was performed in section 2.4. The difference is that the mode numbers of the external $|[\mathbf{9}, \mathbf{1}]\rangle$ state have distinct, odd values n_1 and n_2 satisfying

$$n_1 + n_2 = 2m \tag{2.249}$$

where m is the external wrapping number. For the impurity-conserving channel, we may either place the two intermediate-state impurities on the same string (say string #1), or one on each string. In the former case string #2 is in its vacuum state and necessarily has wrapping number $m_2 = 0$. The level-matching condition for the excited string gives $q_1 + q_2 = m_1$, where the q_i are the internal mode numbers; conservation of wrapping number then gives $m_1 = m$. In the latter case we have $q_1 = m_1$, and $q_2 = m_2$ while $m_1 + m_2 = m$. Thus the two choices for distribution of intermediate state impurities are indistinguishable, both leading to the same condition which is introduced into the amplitudes via the factor $\delta_{q_1, m-q_2}$ where $m = (n_1 + n_2)/2 \in \mathbb{Z}$. We begin with the SVPS result for the H_3 term

$$\begin{aligned}\delta E^{\text{SVPS}}_{H_3} = \frac{2}{r(1-r)} \frac{g_2^2\, \alpha'^2}{64\, \alpha_3^6} \sum_{r_1\, r_2} \sum_{q_1\, q_2} \Bigg[& \left(L^{3\, r_1}_{n_1\, q_1}\right)^2 \left(\widetilde{N}^{3\, r_2}_{n_2\, q_2}\right)^2 + L^{3\, r_1}_{n_1\, q_1} L^{3\, r_2}_{n_1\, q_2} \widetilde{N}^{3\, r_2}_{n_2\, q_2} \widetilde{N}^{3\, r_1}_{n_2\, q_1} \\ & + L^{3\, r_1}_{n_2\, q_1} L^{3\, r_1}_{n_1\, q_1} \widetilde{N}^{3\, r_2}_{n_1\, q_2} \widetilde{N}^{3\, r_2}_{n_2\, q_2} + L^{3\, r_1}_{n_2\, q_1} L^{3\, r_2}_{n_1\, q_2} \widetilde{N}^{3\, r_2}_{n_1\, q_2} \widetilde{N}^{3\, r_1}_{n_2\, q_1} \Bigg] \\ & \times \frac{-\alpha_3\, \delta_{q_1, m-q_2}}{\omega_{n_1} + \omega_{n_2} - \beta^{-1}_{r_1}\omega_{q_1} - \beta^{-1}_{r_2}\omega_{q_2}} + (n_1 \leftrightarrow n_2)\end{aligned} \tag{2.250}$$

where now instead of an integration over a continuous $r \in [0, 1]$, r is fixed at $1/2$. The result is

$$\left(\delta E_{H_3}^{\rm SVPS}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[8\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - \frac{1}{\pi}\left(\frac{13}{4}(n_1^2+n_2^2)+\frac{1}{2}n_1 n_2\right)\lambda'^{5/2} - \frac{1}{\pi^2}\left(\frac{22}{3}(n_1^2+n_2^2)+\frac{4}{3}n_1 n_2\right)\lambda'^{5/2}+\dots\Bigg]. \tag{2.251}$$

The contact term contribution is as follows

$$\delta E_{H_4}^{\rm SVPS} = -\frac{g_2^2\,\alpha'}{16\,\alpha_3^3}\sum_{r_1 r_2}\sum_{q_1 q_2}\left[(K_{-n_1})^2\,(G_{q_1})^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + K_{-n_1}K_{-n_2}\,(G_{q_1})^2\,\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right]\delta_{q_1,m-q_2} +(n_1\leftrightarrow n_2) \tag{2.252}$$

giving

$$\left(\delta E_{H_4}^{\rm SVPS}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[\lambda' + \left(\frac{n_1+n_2}{2} - 4\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^{3/2} - \left(\frac{n_1^2+n_2^2}{4}+2(n_1+n_2)\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^2 - \left(\frac{n_1^3+n_2^3}{2} - \frac{1}{\pi}\left(\frac{13}{8}(n_1^2+n_2^2)+\frac{1}{4}n_2 n_1\right) - \frac{1}{\pi^2}\left(\frac{23}{6}(n_1^2+n_2^2)+\frac{1}{3}n_1 n_2\right)\right)\lambda'^{5/2}+\dots\Bigg]. \tag{2.253}$$

Combining the results we find

$$\boxed{\left(\delta E^{\rm SVPS}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[\lambda' + \left(\frac{n_1+n_2}{2} + 4\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^{3/2} - \left(\frac{n_1^2+n_2^2}{4}+2(n_1+n_2)\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^2 - \left(\frac{n_1^3+n_2^3}{2} + \frac{1}{\pi}\left(\frac{13}{8}(n_1^2+n_2^2)+\frac{1}{4}n_2 n_1\right) + \frac{1}{\pi^2}\left(\frac{7}{2}(n_1^2+n_2^2)+n_1 n_2\right)\right)\lambda'^{5/2}+\dots\Bigg].} \tag{2.254}$$

This result does display a leading agreement with the gauge theory result (2.247). However, it also suffers maximally from half-integer powers of λ'. We will see that the DY vertex will do better, in analogy with the standard case. First, we present the results for the DVPPRT vertex. The expression for the H_3 term is

$$\delta E_{H_3}^{\mathrm{DVPPRT}} = \frac{2}{r(1-r)}\frac{g_2^2\,\alpha'^2}{64\,\alpha_3^6}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\Bigg[\left(\widetilde{L}^{3\,r_1}_{n_1\,q_1}\right)^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + \widetilde{L}^{3\,r_1}_{n_1\,q_1}\widetilde{L}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\widetilde{N}^{3\,r_1}_{n_2\,q_1}$$
$$+ \widetilde{L}^{3\,r_1}_{n_2\,q_1}\widetilde{L}^{3\,r_1}_{n_1\,q_1}\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2} + \widetilde{L}^{3\,r_1}_{n_2\,q_1}\widetilde{L}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_1}_{n_2\,q_1}\Bigg]$$
$$\times\frac{-\alpha_3\,\delta_{q_1,m-q_2}}{\omega_{n_1}+\omega_{n_2}-\beta_{r_1}^{-1}\omega_{q_1}-\beta_{r_2}^{-1}\omega_{q_2}} + (n_1\leftrightarrow n_2) \tag{2.255}$$

with result

$$\left(\delta E_{H_3}^{\mathrm{DVPPRT}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[-2\lambda' + 8\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} + \frac{3}{2}(n_1^2+n_2^2)\lambda'^2$$
$$-\frac{1}{\pi}\left(\frac{21}{4}(n_1^2+n_2^2)+\frac{1}{2}n_1n_2\right)\lambda'^{5/2} - \frac{1}{\pi^2}\left(\frac{38}{3}(n_1^2+n_2^2)-\frac{4}{3}n_1n_2\right)\lambda'^{5/2}+\dots\Bigg] \tag{2.256}$$

while the contact term gives

$$\delta E_{H_4}^{\mathrm{DVPPRT}} = -\frac{g_2^2\,\alpha'}{16\,\alpha_3^3}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\left[(K_{n_1})^2\,(G_{q_1})^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + K_{n_1}K_{n_2}\,(G_{q_1})^2\,\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right]\delta_{q_1,m-q_2}$$
$$+(n_1\leftrightarrow n_2) \tag{2.257}$$

with result

$$\left(\delta E_{H_4}^{\mathrm{DVPPRT}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[\lambda' - \left(\frac{n_1+n_2}{2}+4\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^{3/2}$$
$$-\left(\frac{n_1^2+n_2^2}{4}-2(n_1+n_2)\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^2$$
$$+\left(\frac{n_1^3+n_2^3}{2}+\frac{1}{\pi}\left(\frac{13}{8}(n_1^2+n_2^2)+\frac{1}{4}n_2n_1\right)+\frac{1}{\pi^2}\left(\frac{23}{6}(n_1^2+n_2^2)+\frac{1}{3}n_1n_2\right)\right)\lambda'^{5/2}+\dots\Bigg]. \tag{2.258}$$

Combining these results we obtain the mass-shift for the DVPPRT vertex

$$\boxed{\begin{aligned}\left(\delta E^{\mathrm{DVPPRT}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[&-\lambda' - \left(\frac{n_1+n_2}{2} - 4\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^{3/2} \\ &+ \left(\frac{5}{4}(n_1^2+n_2^2) + 2(n_1+n_2)\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^2 \\ +\Bigg(\frac{n_1^3+n_2^3}{2} - \frac{1}{\pi}\left(\frac{29}{8}(n_1^2+n_2^2)+\frac{1}{4}n_2n_1\right) &- \frac{1}{\pi^2}\left(\frac{53}{6}(n_1^2+n_2^2)-\frac{5}{3}n_1n_2\right)\Bigg)\lambda'^{5/2}+\ldots\Bigg]\end{aligned}} \tag{2.259}$$

which fails to agree with the gauge theory result even at the leading order, as the sign is incorrect. Finally, we compute the extra cross-terms required to assemble the DY result. The H_3 cross-term is given by

$$\begin{aligned}\delta E^{\text{S-DV}}_{H_3} &= 2\frac{\langle H_3^{\mathrm{DVPPRT}}|e\rangle\langle e|H_3^{\mathrm{SVPS}}\rangle}{\Delta E} = \\ &\frac{2}{r(1-r)}\frac{g_2^2\,\alpha'^2}{32\,\alpha_3^6}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\Bigg[\widetilde{L}^{3\,r_1}_{n_1\,q_1}L^{3\,r_1}_{n_1\,q_1}\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + \widetilde{L}^{3\,r_1}_{n_1\,q_1}L^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\widetilde{N}^{3\,r_1}_{n_2\,q_1} \\ &\qquad + \widetilde{L}^{3\,r_1}_{n_2\,q_1}L^{3\,r_1}_{n_1\,q_1}\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2} + \widetilde{L}^{3\,r_1}_{n_2\,q_1}L^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_1}_{n_2\,q_1}\Bigg] \\ &\qquad \times\frac{-\alpha_3\,\delta_{q_1,m-q_2}}{\omega_{n_1}+\omega_{n_2}-\beta_{r_1}^{-1}\omega_{q_1}-\beta_{r_2}^{-1}\omega_{q_2}} + (n_1\leftrightarrow n_2)\end{aligned} \tag{2.260}$$

resulting in

$$\begin{aligned}\left(\delta E^{\text{S-DV}}_{H_3}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[&8\lambda' - 16\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - 4(n_1^2+n_2^2)\lambda'^2 + \frac{1}{\pi}\left(\frac{17}{2}(n_1^2+n_2^2)+n_1n_2\right)\lambda'^{5/2} \\ &+ \frac{20}{\pi^2}(n_1^2+n_2^2)\lambda'^{5/2}+\ldots\Bigg].\end{aligned} \tag{2.261}$$

The contact cross-term is given by

$$\begin{aligned}\delta E^{\text{S-DV}}_{H_4} &= \frac{1}{2}\langle Q_3^{\mathrm{DVPPRT}}|e\rangle\langle e|Q_3^{\mathrm{SVPS}}\rangle = \\ -\frac{g_2^2\,\alpha'}{8\,\alpha_3^3}\sum_{r_1\,r_2}\sum_{q_1\,q_2}&\Bigg[K_{n_1}K_{-n_1}\left(G_{q_1}\right)^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + K_{n_1}K_{-n_2}\left(G_{q_1}\right)^2\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\Bigg]\delta_{q_1,m-q_2} + (n_1\leftrightarrow n_2)\end{aligned} \tag{2.262}$$

with result

$$\left(\delta E^{\text{S-DV}}_{H_4}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[-2\lambda' + 8\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} + \left(n_1^2+n_2^2\right)\lambda'^2 - \left(\frac{1}{\pi}\left(\frac{15}{4}(n_1^2+n_2^2)+\frac{3}{2}n_2 n_1\right)+\frac{1}{\pi^2}\left(\frac{26}{3}(n_1^2+n_2^2)+\frac{8}{3}n_1 n_2\right)\right)\lambda'^{5/2}+\ldots\Bigg]. \tag{2.263}$$

Assembling the final result $\delta E^{\text{DY}} = (\delta E^{\text{DVPPRT}} + \delta E^{\text{S-DV}} + \delta E^{\text{SVPS}})/4$, we find

$$\boxed{\left(\delta E^{\text{DY}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\frac{3}{2}\left[\lambda' - \frac{n_1^2+n_2^2}{3}\lambda'^2 - \frac{(n_1+n_2)^2}{6}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{5/2}+\ldots\right]} \tag{2.264}$$

which we have verified using the so-called "second method" outlined in section 2.4.2. This result matches the leading order gauge theory result (2.247) if we re-scale the undetermined function f (appearing in front of the vertices) by $\sqrt{2/3}$. The result is superior to the SVPS result (2.254) as it does not contain the 3/2's power of λ'. It would be interesting to know whether the λ'^2 term also agrees with gauge theory, however the gauge theory computation of this term has yet to be done.

$k = 3$ Impurity-conserving mass-shift

For the $k = 3$ string, the splitting and level-matching are more involved. There are two distinct cases, the first is when string #1 has $k_1 = 1$. We can then distribute the two intermediate state impurities both on string #1, both on string #2, or one impurity per string (of which there are two equivalent configurations). The next case is when the assignments of light-cone momenta are reversed, so that string #1 has $k_1 = 2$ (and so string #2 has $k_2 = 1$). This just counts the $k_1 = 1$ case again, leading to a factor of two. The level-matching is therefore achieved via the insertion of the following operator

$$r = \frac{1}{3}, \qquad 2\left(\delta^{r_1,1}\delta^{r_2,1}\delta_{q_1,m-q_2} + \delta^{r_1,2}\delta^{r_2,2}\delta_{q_1,2m-q_2} + 2\delta^{r_1,1}\delta^{r_2,2}\delta_{q_1,2(m-q_1)}\right) \tag{2.265}$$

where the intermediate-state impurities have mode-number/string label configurations (q_1, r_1) and (q_2, r_2), and $m = (n_1 + n_2)/3 \in \mathbb{Z}$ is the external state winding number while n_1 and n_2 are integers and not multiples of three.

The expressions given for the $k = 2$ case in the previous subsection are equally valid here, however with the replacement of the $k = 2$ delta function with (2.265). The results are difficult to obtain for high order in λ', and so we present leading order results only. Since the calculations are straightforward, we will be brief and simply state the results

$$\boxed{\left(\delta E^{\mathrm{SVPS}}\right)_{k=3} = \frac{g_2^2\lambda'}{16\pi^2}\left[1+\frac{9}{2}\frac{\left[\cos\left(\frac{\pi n_1}{3}\right)\sin\left(\frac{\pi n_1}{3}\right)-\cos\left(\frac{\pi n_2}{3}\right)\sin\left(\frac{\pi n_2}{3}\right)\right]}{\pi(n_1-n_2)}\right]+\ldots} \tag{2.266}$$

$$\boxed{\left(\delta E^{\mathrm{DVPPRT}}\right)_{k=3} = \frac{g_2^2\lambda'}{16\pi^2}\left[-1+\frac{3}{2}\frac{\left[\cos\left(\frac{\pi n_1}{3}\right)\sin\left(\frac{\pi n_1}{3}\right)-\cos\left(\frac{\pi n_2}{3}\right)\sin\left(\frac{\pi n_2}{3}\right)\right]}{\pi(n_1-n_2)}\right]+\ldots} \tag{2.267}$$

$$\boxed{\left(\delta E^{\mathrm{DY}}\right)_{k=3} = \frac{g_2^2\lambda'}{16\pi^2}\frac{3}{2}\left[1+\frac{3}{2}\frac{\left[\cos\left(\frac{\pi n_1}{3}\right)\sin\left(\frac{\pi n_1}{3}\right)-\cos\left(\frac{\pi n_2}{3}\right)\sin\left(\frac{\pi n_2}{3}\right)\right]}{\pi(n_1-n_2)}\right]+\ldots} \tag{2.268}$$

Comparing with the gauge theory result (2.248), we see that although the dependence on the external mode numbers is of the correct form, the coefficient of the second term is not matched by any of the vertices. Further, the first term of the DVPPRT does not match on account of the sign.

$k = 2$ Four impurity channel mass-shift

We have had success in matching the leading $k = 2$ mass-shift to gauge theory using both the SVPS and DY vertices and the impurity conserving channel. It is therefore interesting to see whether or not a miraculous cancellation appears at the four-impurity channel, such that it leads as λ'^2 or higher. The $k = 2$ setting makes the calculation simpler than it would be for the standard, continuous p^+ case. The reasons for this are as follows.

1. The intermediate strings have only one possible distribution of p^+: each string must have $k = 1$. This gives the same level-matching condition regardless of the distribution of the four impurities amongst the two strings; thus one may be chosen and the result multiplied by 16.

2. The leading λ' term comes only from those expressions containing a double-pole in one of the intermediate mode number sums. This allows us to discard many complicated terms from the calculation.

3. The result will be independent of n_1 and n_2, and therefore just a number. The expressions are therefore simple and easy to manipulate.

The calculation was performed by the author using two methods "in parallel" as a check on the results. The methods used are the standard H_3 and contact term we have been using all along, and the manifestly convergent method (2.139) developed in section 2.3.3. We can relate the quantities appearing in these two methods by exploiting the superalgebra. We have

$$[H_2, Q_3] = [Q_2, H_3]. \tag{2.269}$$

Let $|\phi\rangle = |[\mathbf{9},\mathbf{1}]\rangle$, $|I\rangle =$ be a generic two string intermediate state, $\langle\psi| = \langle\phi|\, Q_2$, and $|\Lambda\rangle = Q_2\,|I\rangle$. Then

$$
\begin{aligned}
\langle\phi|\, H_2\, Q_3 - Q_3\, H_2\,|I\rangle &= \langle\phi|\, Q_2\, H_3 - H_3\, Q_2\,|I\rangle \\
\left(\frac{\omega_n^{(3)}}{-\alpha_3} - \sum_{s=1}^{2}\frac{\omega_{p_s}^{(s)}}{-\beta_s\alpha_3}\right)\langle\phi|Q_3\,|I\rangle &= \langle\psi|\, H_3|I\rangle - \langle\phi|\, H_3\,|\Lambda\rangle
\end{aligned}
\tag{2.270}
$$

therefore

$$
\delta E_1 = \frac{\langle\phi|\langle I|\, Q_3\rangle\ \langle H_3|I\rangle|\psi\rangle}{4\,\Delta E} = \frac{1}{4}\,|\langle\phi|\langle I|Q_3\rangle|^2 + \frac{\langle\phi|\langle I|\, Q_3\rangle\ \langle\phi|\langle\Lambda|H_3\rangle^*}{4\,\Delta E}. \tag{2.271}
$$

Further, we have

$$
\{Q_2, Q_3\} = 4H_3. \tag{2.272}
$$

Taking the expectation value in the same way, we find

$$
\begin{aligned}
\langle\phi|\, Q_2\, Q_3 + Q_3\, Q_2\,|I\rangle &= 4\,\langle\phi|\, H_3\,|I\rangle \\
\langle\psi|Q_3\,|I\rangle + \langle\phi|\, Q_3\,|\Lambda\rangle &= 4\,\langle\phi|\, H_3|I\rangle
\end{aligned}
\tag{2.273}
$$

and therefore

$$
\delta E_2 = \frac{\langle\psi|\langle I|\, Q_3\rangle\ \langle H_3|I\rangle|\phi\rangle}{4\,\Delta E} = \frac{|\langle\phi|\langle I|H_3\rangle|^2}{\Delta E} - \frac{\langle\phi|\langle I|\, H_3\rangle\ \langle\phi|\langle\Lambda|Q_3\rangle^*}{4\,\Delta E}. \tag{2.274}
$$

We have calculated all three terms in (2.271) and in (2.274), for the four impurity channel, at leading order in λ', checking that the two methods give the same result. Many of the matrix elements are to be found in appendix D. The various intermediate states may be classified by the number of α_q^i, $\alpha_q^{i'}$, $\beta_q^{\alpha_1\alpha_2}$, and $\beta_q^{\dot\alpha_1\dot\alpha_2}$ impurities. As an example we will show the δE_1 calculation of the $\alpha'\,\alpha\,\alpha\,\beta$ channel - i.e. one undotted fermion, two bosons from the first $SO(4)$ and one from the second. We begin by finding the $\alpha'\,\alpha\,\alpha\,\beta$ contributions from $\langle H_3|I\rangle|\psi\rangle$, found in (D.36). The only source for a pole in an intermediate state mode number is the Neumann matrix $\widetilde{N}^{3\,r}_{n\,q}$, therefore we ignore any contribution which does not contain this matrix. Further, as per usual, we are only interested in those contributions which do not result in a delta function on the external state's spacetime indices. We find the contribution to be

$$
\begin{aligned}
&\langle Q_2 : \alpha^k_{n_1}\alpha^l_{n_2}|\langle I|\, H_3\rangle = \\
&\qquad \frac{g_2\alpha'}{8\alpha_3^3}\frac{\bar\eta}{\sqrt{2|\alpha_3|}}\sigma^k_{\beta_1\dot\gamma_1}\Omega_{n_1}G_{n_1}\widetilde{N}^{3\,s_2}_{n_2\,p_2}\alpha^{\dagger\,(s_2)\,l}_{p_2}\left[K^{\dot\rho_1\rho_1}\widetilde{K}^{\dot\rho_2\rho_2} + \widetilde{K}^{\dot\rho_1\rho_1}K^{\dot\rho_2\rho_2}\right]Y_{\rho_1\rho_2}\delta^{\dot\gamma_1}_{\dot\rho_1}\epsilon_{\dot\beta_2\dot\rho_2} \\
&\qquad\qquad + (n_1 \leftrightarrow n_2) \\
&= \frac{g_2\alpha'}{8\alpha_3^3}\frac{\bar\eta}{\sqrt{2|\alpha_3|}}\sigma^k_{\beta_1\dot\gamma_1}\sigma^{i_1\,\dot\gamma_1\rho_1}\sigma^{i'\,\rho_2}{}_{\dot\beta_2}\Omega_{n_1}G_{n_1}\widetilde{N}^{3\,s_2}_{n_2\,p_2}G_{p_1}\left[K_{p_3}K_{-p_4} + K_{-p_3}K_{p_4}\right]\alpha^{\dagger\,l}_{p_2}\alpha^{\dagger\,i_1}_{p_3}\alpha^{\dagger\,i'}_{p_4}\beta^{\dagger\,l}_{p_1\;\rho_1\rho_2} \\
&\qquad\qquad + (n_1 \leftrightarrow n_2).
\end{aligned}
\tag{2.275}
$$

The contribution to the $\alpha'\,\alpha\,\alpha\,\beta$ channel from $\langle\phi|\langle I|\,Q_3\rangle$ is read-off from (D.31), it is

$$\left(\langle\alpha^i_{n_1}\alpha^j_{n_2}|\langle I|\,Q_3\rangle\right)^\dagger = \frac{g_2\overline{\eta}}{4\alpha_3^3}\sqrt{\frac{\alpha'\kappa}{2}}(-i)\,\sigma^{j'}_{\lambda_2\dot\lambda_2}G_{q_4}\epsilon^{\dot\lambda_2\dot\beta_2}K_{-q_3}\widetilde{N}^{3\,r_1}_{n_1\,q_1}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\alpha^{j'}_{q_3}\alpha^i_{q_1}\alpha^j_{q_2}\beta^{\beta_1\lambda_2}_{q_4}. \quad (2.276)$$

The next step is to calculate

$$\left(\langle\alpha^i_{n_1}\alpha^j_{n_2}|\langle I|\,Q_3\rangle\right)^\dagger\langle Q_2:\alpha^k_{n_1}\alpha^l_{n_2}|\langle I|\,H_3\rangle = -\frac{g_2^2\alpha'^{3/2}}{128\alpha_3^6}(-\alpha_3)\left(2\delta^{ik}\delta^{jl}\right)\left(2\delta^{i'j'}\delta^{i'j'}\right)\Omega_{n_1}G_{n_1}$$
$$\times K_{-q_3}\left(G_{q_4}\right)^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2\widetilde{N}^{3\,r_1}_{n_1\,q_1}\left[K_{q_1}K_{-q_3}+K_{-q_1}K_{q_3}\right]+(n_1\leftrightarrow n_2). \quad (2.277)$$

Finally, we must level-match and sum. This is accomplished via

$$\sum_{r_1,r_2,r_3,r_4=1}^{2}\;\sum_{\substack{q_1,q_2,q_3,q_4\\ \sum q_i=m}} = 16\sum_{\substack{q_1,q_2,q_3,q_4\\ \sum q_i=m}} \quad (2.278)$$

reflecting the fact that all distributions of intermediate-state impurities over the internal strings are equivalent. The factor of $\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2$ in (2.277) plays a very important rôle. It provides a double pole in the sum over q_2, fixing it to $n_2/2$ and causing $\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2$ to evaluate to $\frac{1}{2}$ in the large-μ limit. Since the remaining mode numbers will effectively be order-μ, one can simply set n_1 and n_2 to zero, leaving a sum over two mode-numbers, q_1 and q_3 say, while $q_4=-(q_1+q_3)$. Taking the large-μ limit of the remaining expressions, we find

$$\delta E_1 = \frac{g_2^2\lambda'}{16\pi^4}\int_{-\infty}^{\infty}dq_1\int_{-\infty}^{\infty}dq_3\frac{\left(\Lambda_3^+ + \Lambda_3^-\right)\Lambda_1^+\left[\Lambda_1^+\Lambda_3^+ - \Lambda_1^-\Lambda_3^-\right]}{q_1\omega_1\omega_3\omega_4\left(1-\omega_1-\omega_3-\omega_4\right)} \quad (2.279)$$

where we have scaled $\mu\alpha_3$ out of all quantities so that

$$\omega_i=\sqrt{q_i^2+1},\qquad \Lambda_i^+=\sqrt{\omega_i+1},\qquad \Lambda_i^-=e(q_i)\sqrt{\omega_i-1}. \quad (2.280)$$

Unfortunately (2.279) is as far as we can go, integrals of this form do not have closed analytical solutions. However, we can still calculate the total four-impurity channel shift and express it in terms of integrals of this form. We have done this, and verified our results as indicated previously, by ensuring that (2.271) and (2.274) are satisfied for each channel (i.e. combinations of impurities), and finally that both the standard method and $(\delta E_1 + \delta E_2)/4$ give the same result for the complete four-impurity mass-shift. The cancellation of divergences discussed in section 2.3 are found explicitly; this is a confirmation of the work in that section.

A complete presentation of the calculation would fill many pages and we will opt not to do this. The result is however, that neither the SVPS, DVPPRT, nor the DY vertices give a zero result. These vertices contribute to the leading λ' order for the four-impurity channel. The results may be expressed by approximate numerical results for the integrals

$$\boxed{\delta E^{\text{4-imp.}}_{k=2} = \frac{g_2^2\lambda'}{\pi^4} \times \begin{cases} -0.68 & \text{SVPS} \\ 0.29 \;\; (0.22 \;\; L.R.) & \text{DY} \\ -0.42 & \text{DVPPRT} \end{cases}} \qquad (2.281)$$

where we have indicated in brackets the result of a suggested extension of the DY vertices proposed by Lee and Russo [109], affecting only the impurity non-conserving channels.

2.5.3 Discussion

The DLCQ light-cone string field theory has been investigated at the impurity conserving level for the $k = 2$ and $k = 3$ two impurity external state. Further the four-impurity channel has also been investigated for the $k = 2$ state. The results for $k = 2$ are inconclusive. We have available only an external mode number independent prediction from gauge theory (2.247) to compare to. The impurity conserving channel gives such a number for any of the three vertices considered (2.254), (2.259), (2.264). The four-impurity channel also contributes at this leading order (2.281). The total shift from the impurity-conserving and four-impurity channel is negative for SVPS and DVPPRT, which is a mismatch with gauge theory, but since we have no evidence of a truncation of λ' terms above four impurities, higher channels may correct this. It is reassuring that the DY result is free of half-powers of λ' at the impurity-conserving level; however the higher orders of the four-impurity channel could easily contain half-integer powers; our analysis was only able to capture the leading term.

The results for $k = 3$ (2.266 - 2.268) fail to reproduce the gauge theory result (2.248). The dependence on the external mode numbers is correct, it is the coefficients which are mismatched. What effect higher impurity channels may have on these results remains a mystery.

The broad outlines of the gauge theory results are captured here - truncation of the $k = 2$ spectrum, protection of the $k = 1$ spectrum, and the absence of corrections when the external mode numbers are multiples of the external light-cone momenta. The general form of corrections also seems correct, however the precise details continue to be lacking. The main issue is the effect of higher impurity channels. Until these can be brought under control, the validity of the vertices cannot truly be known.

2.6 Conclusions

In the plane-wave limit, the AdS/CFT correspondence stands the best chance of being systematically tested beyond the classical (i.e. planar) level. Light-cone string field theory on the plane-wave background is the tool for carrying out such tests. As it stands, the correct form of the string interaction vertices is ambiguous. Various proposals are put forward, but at the base of all of them is the fundamental construction on the foundation of (super)-locality: the strings must touch (in superspace) where they interact. Unfortunately symmetry alone is not enough to completely fix the interactions. There are two main issues as regards the light-cone string field theory on the plane-wave 1) The lack of a construction for the quartic supercharge Q_4, and 2) The lack of tools to analyze the higher impurity channels. Without these ingredients, the validity of the proposed vertices will likely remain unknown.

It is troublesome that a correspondence conjectured to be valid on the basis of symmetries fails to be tested due to a lack of (string theory) information beyond those symmetries. Indeed, we do not have a first principles approach to constructing *the* light-cone string field theory; and so attempting to match gauge theory results takes on an air of predetermined conclusions. On the other hand many features of the gauge theory treatment are manifested in the light-cone string field theory and the question of agreement essentially comes down to one of coefficients.

The structure of this string field theory deserves to be explored further. One would not be too surprised to find a cancellation mechanism limiting the order of the results in λ' as the number of intermediate state impurities is increased. Further, a cancellation mechanism for the half-powers of λ', as was shown in this chapter for $\sqrt{\lambda'}$, seems possible and worth looking for. An explicit construction involving a non-zero Q_4 would also go a long way in elaborating the theory. Testing the AdS/CFT correspondence at the "quantum" level, that is, the non-planar/string-loop level, remains one of the most important pursuits in fleshing-out and comprehending the duality.

Chapter 3

Free energy and phase transition of the matrix model on a plane-wave

Double, double toil and trouble;
Fire burn, and caldron bubble.
— Shakespeare's *Macbeth*, Act IV, Scene 1

In section 1.4.6 we mentioned that 11-dimensional supergravity (1.49) plays a privileged rôle. Eleven is the maximum spacetime dimension before massless particles of spin greater than 2 are introduced by supergravity. The lower dimensional supergravities are (essentially) derivable via dimensional reduction from this master theory. Superstring theory is a quantization of 10-dimensional gravity; finding a supersymmetric quantization of 11-dimensional gravity might then produce a master theory from which all string theories are derivable. It was in this effort that "M-theory" or "Matrix" theory was developed. The path was to attempt the quantization of a membrane (a 2-spatial dimensional object) in an 11-dimensional target space. Working in the light-cone gauge and promoting spatial worldvolume coordinates to matrices, a regularization or discretization was achieved, resulting in a theory of $N \times N$ matrices which depend on a single time-like parameter. This matrix quantum mechanics was shown by Banks, Fischler, Shenker, and Susskind (BFSS) [110] to also describe a collection of N D0-branes in type-IIA superstring theory. It was then found that various classical and quantum mechanical processes in 11-d supergravity were captured by the matrix model, leading to the conjecture that the full second-quantized theory containing 11-d SUGRA as its low-energy limit was encoded by the matrix model. A more ambitious proposal is that the BFSS matrix model describes a *master* theory containing within it, as limits, all known string theories as well as 11-d SUGRA, see figure 3.1.

The BFSS matrix model suffers from the drawback that it does not contain a perturbative coupling constant. The plane-wave background (2.1) introduced by Berenstein, Maldacena, and Nastase [61] also has a cousin in 11-dimensional SUGRA, and the membrane can be quantized in the presence of this background. Alternatively, one may consider the collection of N D0-branes on the 10-dimensional type-IIA plane-wave. These approaches both lead to the plane-wave matrix model which does have a perturbative coupling [117]. Essentially, this matrix model (and the BFSS model) is a 1-dimensional gauge theory whose large-N limit corresponds to the low-energy 11-dimensional SUGRA limit. In this sense, it is a manifestation of a gauge/gravity duality like the AdS/CFT correspondence. In the standard AdS/CFT duality, considering the CFT at finite temperature is dual to a gas of gravitons in the AdS space, characterized by the same temperature. As the temperature is raised, the AdS space undergoes a phase transition leading to the production of a large black-hole, an object which is thermally stable. This is known as the Hawking-Page phase transition [111]. The transition on the gauge theory side is conjectured to be the analogue of the deconfine-

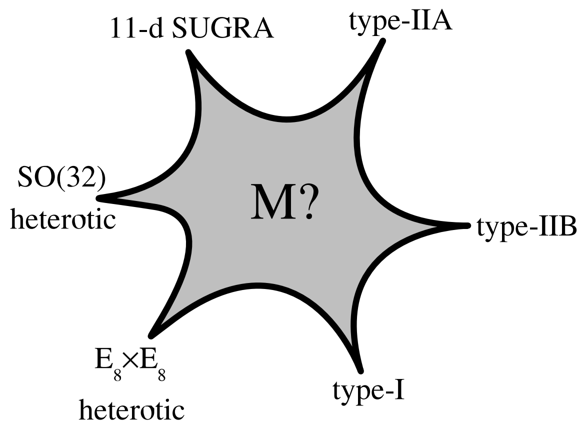


Figure 3.1: The five types of string theories: type-IIA, type-IIB, type-I, and the two heterotic string theories are related via various dualities. The low-energy limit of the BFSS matrix model, which may be understood as a collection of D0-branes in type-IIA string theory, gives 11-dimensional supergravity. Could this matrix model also represent a master theory from which all string theories arise as limits?

ment transition in QCD [112]. At low energies, the degrees of freedom are singlets of the gauge group $SU(N)$, and so the free energy is of order one. As the temperature is raised, charged states are liberated, so that at high enough temperature every possible state is excited. In this phase the free energy scales as N^2, the total number of fundamental excitations of the theory. As we will see, the plane-wave matrix model also shares a deconfinement transition. The dual gravity interpretation, however, is less clear. Determining the order of this transition is therefore an interesting endeavour, as it should shed some light on the dual process. Notwithstanding that, it is of general interest to understand deconfinement transitions wherever they arise, as this information should help us to eventually understand the QCD deconfinement transition, a subject of paramount importance in physics and cosmology.

3.1 M-theory and the BFSS matrix model

In section 1.4.6, we showed how type-IIA supergravity could be derived from 11-dimensional supergravity via dimensional reduction. When a theory is dimensionally reduced, the extra dimensions are taken to be compact with radius R. This gives the familiar Kaluza-Klein mechanism, where momentum in the compact direction becomes mass in the dimensionally reduced theory. The mass comes in units of R^{-1}, and so as $R \to 0$, the zero-momentum modes become decoupled from the infinitely more massive Kaluza-Klein states, which can be ignored. Beginning with type-IIA string theory, we can actually follow this process in reverse, and watch while this theory *grows* an extra dimension, becoming a theory whose low-energy limit is 11-d SUGRA, i.e. M-theory. The trick is to consider the D0-branes of type-IIA superstring theory. These objects have a mass given by

$$\tau_0 = \frac{1}{g_s\sqrt{\alpha'}} \tag{3.1}$$

where g_s is the string coupling. These objects are charged under the Ramond-Ramond vector potential A_1, see (1.50). We discussed in section 1.5.1 that parallel D-branes do not interact as a consequence of their gravitational attraction balancing their form-field repulsion exactly. A stable ground state of a system of parallel Dp-branes was then to have them coincident, as excitations consisting of open strings stretched between separated pairs would tend to pull them together. A collection of point-like objects are always parallel, and so a coincident arrangement of n D0-branes counts simply n times the D0-brane mass

$$\tau_n = n\tau_0 = \frac{n}{g_s\sqrt{\alpha'}}. \tag{3.2}$$

This is immediately reminiscent of a tower of Kaluza-Klein states on a compact direction of radius $R = g_s\sqrt{\alpha'}$. Indeed, as $g_s \to \infty$, the spectrum (3.2) becomes continuous, $R \to \infty$, and type-IIA string theory grows a new decompactified direction out of its non-perturbative, point-like D0-branes. This strong-coupling limit of type-IIA superstring theory, whatever its true description may be, is given the name "M-theory". At the supergravity level, M-theory is just 11-dimensional supergravity, whose action is (1.49).

The action of N D0-branes may be derived as a dimensional reduction of 10-dimensional supersymmetric Yang-Mills theory with gauge group $SU(N)$ to 0+1-dimensions. Nine of the ten components of the gauge field A_μ become the scalar fields X^I, while the remaining gauge field is zero-dimensional and is called A_0

$$S = \frac{1}{2R}\,\mathrm{Tr}\int dt\left[D_0X^ID_0X^I + i\,\theta^T D_0\,\theta + \frac{1}{2}\left[X^I, X^J\right]^2 - \theta^T\gamma_I\left[\theta, X^I\right]\right] \tag{3.3}$$

where $D_0 = \partial_t - i[A_0, \ldots]$, the fermionic superpartners θ have been included, and all fields are $N\times N$ matrices. The scalars X^I have a very pretty interpretation. We know from section 1.4.7 that the VEV's of these fields describe the transverse shape of a general D-brane. In fact, here, for D0-branes, all spatial directions are transverse. The "position" of the D0-branes may become non-commutative or "fuzzy". The lowest energy configuration is to take the $\langle X^I\rangle$ constant, and have the commutator vanish, thus allowing them to be simultaneously diagonalized. The eigenvalues are precisely the positions of the N D0-branes. Turning on off-diagonal elements of $\langle X^I\rangle$ gives a non-commutative geometry, where the "positions" of the branes are matrix valued. In two important works, [110] and [113], convincing arguments were given that (3.3) indeed describes the discrete light-cone quantization or DLCQ of M-theory. The infinite N limit should then correspond to decompactified M-theory viewed in the infinite momentum frame [114]. Perhaps the most convincing evidence is that 11-dimensional supergravity scattering amplitudes are readily retrieved using (3.3), as are the extended objects of 11-d SUGRA (see [115] for a review). The significance of the BFSS model is that (3.3) was obtained previous to those authors' work, in a very different context. In the 1980's there was a campaign to attempt the quantization of 11-dimensional gravity via a two-dimensional membrane, in much the same way that a one-dimensional string led to the quantization of 10-dimensional gravity [116] (see [115] for a modern review and more references). This work led to the promotion of the membrane embedding functions to matrices, in order to provide a regularization to the theory. The action was then found to be precisely (3.3).

The membrane/BFSS theory has flat-directions (those in which the commutator vanishes), leading to a continuous spectrum. Prior to BFSS this was interpreted as an instability in the dynamics of the membrane. Adding a long spike, of vanishing area, to a membrane incurs a vanishing energy cost. A "sea-urchin" picture of the membrane then emerges, with large and wild fluctuations in membrane shape which cost nearly no energy, leading to a continuous spectrum. This instability was a stumbling block for membrane research, and stymied its progress. BFSS provided a natural interpretation for this continuous spectrum. The theory ought to be considered *second* quantized, as it is capable of describing multi-particle states (i.e. multiple D0-branes); ergo a continuous spectrum. Indeed, as we have mentioned above, BFSS showed that (3.3) was capable of describing the scattering of multi-particle states in 11-d SUGRA.

The continuous spectrum of the BFSS model, though turned from a liability to an asset, still makes calculations challenging compared to a model with a discrete spectrum. The other drawback of (3.3) is that it has no tunable coupling constant. The coupling R is essentially the 11-dimensional Newton's constant, leading to a rather peculiar quantum-classical correspondence. The non-linear terms of 11-dimensional Einstein gravity are reproduced through quantum loop corrections stemming from (3.3). The full classical 11-d gravity therefore requires the all-loop results of the matrix quantum mechanics. For a general process, these loop corrections are not perturbative; indeed there is no sense in which R is small.

3.2 The plane-wave matrix model

In the seminal work by Berenstein, Maldacena, and Nastase [61], a deformation of the BFSS model was given which may be understood as the action of N D0-branes on the type-IIA plane-wave background, or equivalently as the quantization of the supermembrane in an 11-d SUGRA plane-wave [117]. The trough of the plane-wave background (see figure 2.6) causes the previously flat directions to become massive, leading to a discrete spectrum, while the parameter μ leads to a tunable coupling constant. The plane-wave matrix model thus cures the two drawbacks of the BFSS model, and presents itself as an instance of M-theory which readily lends itself to exploration. The 11-dimensional plane-wave can be obtained via a Penrose limit either of $AdS_4 \times S^7$ or $AdS_7 \times S^4$, both maximally symmetric solutions of 11-dimensional supergravity. The result is a plane-wave with different masses for three of the transverse directions as compared to the remaining six

$$\begin{aligned} ds^2 = -2dx^+dx^- + dx^I dx^I - \left[\left(\frac{\mu}{3}\right)^2 x^{\bar{a}}x^{\bar{a}} + \left(\frac{\mu}{6}\right)^2 x^i x^i\right] dx^+dx^+ \\ F_{123+} = \mu \end{aligned} \tag{3.4}$$

where $I = 1, \ldots, 9$, $\bar{a} = 1, \ldots 3$, and $i = 4, \ldots, 9$. The action of the plane-wave matrix model is then given by [61], [117]

$$
\begin{aligned}
S = \frac{1}{2R}\int d\tau\, \mathrm{Tr}\Bigg(& DX^{\bar{a}}DX^{\bar{a}} + DX^{i}DX^{i} + i\psi^{\dagger I\alpha}D\psi_{I\alpha} \\
& - \left(\frac{\mu}{3}\right)^2 (X^{\bar{a}})^2 - \left(\frac{\mu}{6}\right)^2 (X^{i})^2 - \frac{\mu}{4}\psi^{\dagger I\alpha}\psi_{I\alpha} \\
& + \frac{R^2}{2}[X^{\bar{a}}, X^{\bar{b}}]^2 + R^2[X^{\bar{a}}, X^{i}]^2 + \frac{R^2}{2}[X^{i}, X^{j}]^2 - i\mu\frac{2R}{3}\epsilon_{\bar{a}\bar{b}\bar{c}}X^{\bar{a}}X^{\bar{b}}X^{\bar{c}} \\
- R\,\psi^{\dagger I\alpha}\sigma^{\bar{a}\beta}_{\alpha}\left[X^{\bar{a}}, \psi_{I\beta}\right] & + \frac{R}{2}\epsilon_{\alpha\beta}\psi^{\dagger\alpha I}\mathsf{g}^{i}_{IJ}\left[X^{i}, \psi^{\dagger\beta J}\right] - \frac{R}{2}\epsilon^{\alpha\beta}\psi_{\alpha I}(\mathsf{g}^{i\dagger})^{IJ}\left[X^{i}, \psi_{\beta J}\right]\Bigg)
\end{aligned}
\tag{3.5}
$$

where all variables transform in the adjoint representation of the gauge group $X^i \to UX^iU^\dagger$, etc. The time derivatives are covariant, $D = \partial_\tau - i\,[A, ...]$ with an $N \times N$ Hermitian gauge field A. The fermions have 8 complex components with $I, J = 1, \ldots, 4$ and $\alpha, \beta = 1, 2$. The spin matrix has the property $\mathsf{g}^i(\mathsf{g}^j)^\dagger + \mathsf{g}^j(\mathsf{g}^i)^\dagger = 2\delta^{ij}\mathbf{1}_{4\times4}$. $\epsilon_{\alpha\beta}$ and $\epsilon_{\bar{a}\bar{b}\bar{c}}$ are antisymmetric tensors.

The classical supersymmetric vacuua of (3.5), and the perturbation theory about those vacuua, were discovered by Dasgupta, Sheikh-Jabbari, and Van Raamsdonk [117]. They noted that the bosonic potential is given by

$$
V = \frac{R}{2}\,\mathrm{Tr}\left[\left(\frac{\mu}{3R}X^{\bar{a}} + i\epsilon^{\bar{a}\bar{b}\bar{c}}X^{\bar{b}}X^{\bar{c}}\right)^2 - \frac{1}{2}\left[X^i, X^j\right]^2 - \left[X^i, X^{\bar{a}}\right]^2 + \left(\frac{\mu}{6R}\right)^2 X^iX^i\right] \tag{3.6}
$$

where, for supersymmetric solutions, each term must vanish independently. The solutions are simple and beautiful

$$
X^{\bar{a}} = \frac{\mu}{3R}J^{\bar{a}}, \qquad X^i = 0 \tag{3.7}
$$

where $J^{\bar{a}}$ are an N-dimensional representation of $SU(2)$

$$
[J^{\bar{a}}, J^{\bar{b}}] = i\epsilon^{\bar{a}\bar{b}\bar{c}}J^{\bar{c}}. \tag{3.8}
$$

The M-theory interpretation of these vacuua was given in [117], and in a subsequent paper [118]. The extended objects of 11-dimensional supergravity are of two varieties. The action (1.49) contains a three-form potential indicating that objects with 3-dimensional or 6-dimensional worldvolumes can couple to it electrically or magnetically, respectively. These are the membranes or “M2-branes” and fivebranes or “M5-branes” of the theory. A general N-dimensional representation of $SU(2)$ has a block-diagonal structure where the size of the blocks is given by a partition $\{N_1, \ldots, N_k\}$ of N, i.e. $\sum_{i=1}^{k} N_i = N$. Let $N_1 \geq N_2 \geq \ldots \geq N_k$, then the partition may be represented by a Young tableau, see figure 3.2, with k columns whose depths are given by $\{N_1, \ldots, N_k\}$. These are naturally interpreted as a collection non-commutative “fuzzy-spheres”, which approach, in the large-N limit, a collection of spherical M2-branes with radii

$$
r_i = \sqrt{\frac{1}{N_i}\,\mathrm{Tr}\,X_i^2} = \frac{\mu N_i}{6R} = \frac{\mu\, p_i^+}{6} \tag{3.9}
$$

Figure 3.2: The vacuua of the plane-wave matrix model (3.5) are given by N-dimensional representations of $SU(2)$. These may in turn be pictured as Young tableaux. A given representation may be interpreted as a collection of membranes (shown on the left) where each column corresponds to a single membrane whose radius is proportional to its size N_i; or as a collection of five-branes, where the rôles of column and row are reversed (shown on the right).

where we have indicated the blocks of $X^{\bar{a}}$ via the index i, and have used the fact that $J^{\bar{a}}J^{\bar{a}} = \mathbf{1}_{N\times N}(N^2-1)/4$. We have also noted that N_i/R is to be interpreted as the amount of light-cone momentum p^+ in the DLCQ of M-theory. However, if every vacuum is interpreted as M2-branes, this leaves the question of how the M5-brane vacuua are encoded in the theory. The work of Maldacena, Van Raamsdonk, and Sheikh-Jabbari [118] cleared-up this riddle. For finite N, a given vacuum is ambiguous. In addition to the M2-brane interpretation, it may also be viewed as a collection of n five-branes, where n is the size of the largest irreducible representation (depth of the deepest column in the Young tableau). The number of units of p^+ (and therefore, in the appropriate limit, the radius) of the five-branes are then given by lengths $\{M_1, \ldots, M_n\}$ of the rows in the Young tableau, see figure 3.2. In other words, the number of units of p^+ of the i-th five-brane is given by M_i, the number of irreducible representations of size greater than or equal to i. These interpretations are disambiguated via different large-N limits. To obtain a classical configuration of M2-branes, one takes all $N_i \to \infty$, while keeping k fixed. The classical M5-branes are obtained by taking all $M_i \to \infty$, while keeping n fixed. This corresponds to having an infinite number of repetitions of each of the k irreducible representations, while keeping the sizes of those representations fixed. The M2-brane limit is just the opposite: infinite-sized representations with fixed repetitions. The trivial vacuum, $X^{\bar{a}} = 0$, being N copies of the trivial one-dimensional representation of $SU(2)$, corresponds to a single five-brane whose (one-loop corrected) radius is [118]

$$r_5 = \sqrt{\frac{1}{N}\,\mathrm{Tr}\,X^2} = \sqrt{\frac{18N^2}{\mu p^+}}. \tag{3.10}$$

The coupling constant arising from perturbation theory about the M2-brane and M5-brane vacuua (i.e. at large-N) are different. The representations of $SU(2)$ break the $U(N)$ gauge symmetry of (3.5) down to a residual $U(n_k)$ symmetry, where n_k is the number of repetitions of the representation with size k. The M2-brane limit has finite n_k and the coupling constant is

$$g_{\text{eff}} = \left(\frac{3R}{\mu}\right)^{\frac{3}{2}}. \tag{3.11}$$

The M5-brane limit, due to its enhanced gauge symmetry, picks-up factors of $n = \sum n_k$ in index loops, leading to a 't Hooft coupling

$$\lambda_{\text{eff}} = g_{\text{eff}}^2 n = \left(\frac{3R}{\mu}\right)^3 n. \tag{3.12}$$

In the next section, we will present work of the author of this thesis concerning the thermodynamics of the plane-wave matrix model about the single five-brane vacuum.

3.3 Free energy and phase transition in the single five-brane vacuum

This section is a presentation of the author's original work published in arXiv:hep-th/0409318 [120].

The thermodynamics of the plane-wave matrix model was investigated by Furuchi, Schreiber, and Semenoff in [121]. They discovered that the theory expanded about the five-brane vacuua demonstrates a first-order phase transition (à la Gross-Witten [122]) corresponding to the deconfinement of the plane-wave matrix model. This transition was found to be unique to the five-brane vacuua; the membrane vacuua do not demonstrate a phase transition. They showed that matrix models generally posses Hagedorn transitions [119], and associated this first-order transition with the Hagedorn temperature of M-theory in the five-brane background. That analysis was based an a one-loop calculation of the effective action expanded about the single five-brane vacuum. It was therefore important to understand whether or not the first-order nature of the transition remained at higher loop-order. As we will describe, the two-loop effective action is not sufficient to answer this question, the three-loop effective action (or at least portions thereof) must be obtained. This calculation was undertaken by the author of this thesis and his collaborators in [120], where it was found that the transition remains first order. In the following subsections the details of this work will be presented.

3.3.1 Introduction

The thermodynamics of a quantum field theory (see [123] for a discussion) are investigated by considering a Euclideanized path-integral in which the time is compactified on a circle of circumference $\beta \sim T^{-1}$, where T is the temperature of the resulting ensemble. As an example, consider a quantum field theory of a single scalar field $\phi(\vec{x}, t)$. The transition amplitude between two states $|\phi_0\rangle$ (at time $t = 0$) and $|\phi_1\rangle$ (at time $t = t'$) defined by

$$\begin{aligned}\phi(\vec{x}, 0)|\phi_0\rangle &= \phi_0(\vec{x})\,|\phi_0\rangle \\ \phi(\vec{x}, 0)|\phi_1\rangle &= \phi_1(\vec{x})\,|\phi_1\rangle\end{aligned} \tag{3.13}$$

is given in the path-integral formalism by (see [28], pg. 282)

$$\langle\phi_1|e^{-iHt'}|\phi_0\rangle = \int_{\substack{\phi(\vec{x},0)=\phi_0(\vec{x})\\ \phi(\vec{x},t')=\phi_1(\vec{x})}} [d\phi] \exp\left(i\int_0^{t'} dt\,\mathcal{L}[\phi(\vec{x},t)]\right) \tag{3.14}$$

where $\mathcal{L}$ is the Langrangian of the system. The partition function for an ensemble defined by the (inverse) temperature β is given by

$$Z = \mathrm{Tr}\, e^{-\beta H} = \sum_\phi \langle\phi|e^{-\beta H}|\phi\rangle \tag{3.15}$$

where $\{|\phi\rangle\}$ is a complete set of states spanning the configuration space of the theory. Thus by taking $it = \tau$, and by setting $it' = \beta$, we find that

$$Z = \sum_\phi \langle\phi|e^{-\beta H}|\phi\rangle = \int_{\phi(\vec{x},\tau)=\phi(\vec{x},\tau+\beta)} [d\phi] \exp\left(\int_0^\beta d\tau\,\mathcal{L}[\phi(\vec{x},\tau)]\right) \tag{3.16}$$

where now the functional integration proceeds over the space of periodic fields $\phi(\vec{x},\tau) = \phi(\vec{x},\tau+\beta)$. For fermions, a similar treatment shows that anti-periodic boundary conditions must be imposed, i.e. $\psi(\tau) = -\psi(\tau+\beta)$. The reason may be traced back to the grassman nature of the fermionic fields. The Euclidean action is then defined as $S_E \equiv -\int_0^\beta d\tau\,\mathcal{L}(\tau)$, so that, for example, the partition function for the plane-wave matrix model is, schematically

$$Z = \int [d\psi]\,[dX]\,\frac{[dA]}{\text{gauge orbits}} e^{-S_E}. \tag{3.17}$$

The free energy is then given simply by $F = -S_E$.

As mentioned at the beginning of this chapter, we will find a “deconfinement” transition in this theory. The concept of confinement is usually associated with spatial separation of quarks in QCD, however the plane-wave matrix model has no spatial dimensions, and so the concept of confinement in this context must be clarified. In a confined phase, all states are singlets of the gauge group. The number of such states is order 1 as compared to the rank N of the gauge group. In this phase we therefore expect that the free energy would not scale with N. In the deconfined phase, the singlet states decompose into liberated, charged states, of which there are as many as the number of elements in the group, i.e. $\sim N^2$. We therefore expect to find

$$\begin{aligned} \lim_{N\to\infty} \frac{F}{N^2} &= 0 \qquad \text{confined} \\ \lim_{N\to\infty} \frac{F}{N^2} &\neq 0 \qquad \text{deconfined.} \end{aligned} \tag{3.18}$$

It requires an infinite amount of energy to insert a charged, fundamental particle, i.e. a quark, into the confined phase. In the deconfined phase, this chemical potential is finite, a reflection of the fact that the quarks are liberated. The difference in the free energy when a quark is added is given by [124, 125]

$$F_q[T] - F_0[T] = -T \ln\langle P\rangle, \qquad P = \frac{1}{N}\,\mathrm{Tr}\left(e^{i\oint d\tau\, A}\right) \tag{3.19}$$

where P, the Wilson loop about the Euclidean time circle, is known as the Polyakov loop. One then has that

$$\begin{aligned} \langle P\rangle &= 0 \qquad \text{confined} \\ \langle P\rangle &\neq 0 \qquad \text{deconfined.} \end{aligned} \tag{3.20}$$

The Polyakov loop is therefore an order parameter for the deconfinement transition. We will proceed by calculating the effective action for the Polyakov loop, in order to determine the critical temperature, and the order of the phase transition in the plane-wave matrix model.

3.3.2 Gauge fixing and 1-loop results

The gauge fixing and 1-loop effective action was worked out in [121]. We provide a summary of these results, taken directly from [120].

The partition function is given by the functional integral

$$Z = \int [dA][dX^i][d\psi]e^{-\int_0^\beta d\tau L[A,X^i,\psi]} \tag{3.21}$$

where L is the Euclidean time Lagrangian

$$\begin{aligned} L = &\frac{1}{2R}\,\mathrm{Tr}\left(DX^iDX^i + DX^{\bar a}DX^{\bar a} - \psi^{\dagger I\alpha}D\psi_{I\alpha}\right) \\ &+\frac{1}{2R}\,\mathrm{Tr}\left(\left(\frac{\mu}{3}\right)^2 (X^{\bar a})^2 + \left(\frac{\mu}{6}\right)^2 (X^i)^2 + \frac{\mu}{4}\psi^{\dagger I\alpha}\psi_{I\alpha} + i\mu\frac{2R}{3}\epsilon_{\bar a\bar b\bar c}X^{\bar a}X^{\bar b}X^{\bar c}\right. \\ &\quad + R\,\psi^{\dagger I\alpha}\sigma_\alpha^{\bar a\,\beta}[X^{\bar a},\psi_{I\beta}] - \frac{R}{2}\epsilon_{\alpha\beta}\psi^{\dagger\alpha I}\mathrm{g}^i_{IJ}[X^i,\psi^{\dagger\beta J}] + \frac{R}{2}\epsilon^{\alpha\beta}\psi_{\alpha I}(\mathrm{g}^{i\dagger})^{IJ}[X^i,\psi_{\alpha J}] \\ &\quad \left. -\frac{R^2}{2}[X^i,X^j]^2 - \frac{R^2}{2}[X^{\bar a},X^{\bar b}]^2 - R^2[X^{\bar a},X^i]^2\right) \end{aligned} \tag{3.22}$$

The bosonic and fermionic variables have periodic and antiperiodic boundary conditions, respectively

$$A(\tau+\beta) = A(\tau) \quad , \quad X^i(\tau+\beta) = X^i(\tau) \quad , \quad \psi(\tau+\beta) = -\psi(\tau).$$

Since the boundary conditions for fermions and bosons are different, supersymmetry is broken explicitly. Of course this is expected at finite temperature where bosons and fermions have different thermal distributions. Supersymmetry is restored in the zero temperature limit. We will see the results of this explicitly in the following.

To begin, we must fix the gauge. It is most convenient to use the gauge freedom to make the variable A static and diagonal,

$$\frac{d}{d\tau}A_{ab} = 0 \quad , \quad A_{ab} = A_a\delta_{ab}$$

Once this is done, the remaining degrees of freedom of A are the time-independent diagonal components, A_a. We shall see that they eventually appear in the form $\exp(i\beta A_a)$.

The Faddeev-Popov determinant for the first of these gauge fixings is[1]

$$\det{}'\left(-\frac{d}{d\tau}\left(-\frac{d}{d\tau}+i(A_a-A_b)\right)\right)=\det{}'\left(-\frac{d}{d\tau}\right)\det{}'\left(-\frac{d}{d\tau}+i(A_a-A_b)\right) \tag{3.23}$$

where the boundary conditions are periodic with period β. The prime means that the zero mode of time derivative operating on periodic functions is omitted from the determinant. Once the gauge field is time-independent, we do the further gauge fixing which makes it diagonal. The Faddeev-Popov determinant for diagonalizing it is the familiar Vandermonde determinant,

$$\prod_{a\neq b}|A_a-A_b|.$$

This is also just the factor that the time independent zero mode would contribute to the second of the determinants in (3.23). Including it gives the determinant

$$\prod_{a\neq b}\det{}'\left(-\frac{d}{d\tau}\right)\det\left(-\frac{d}{d\tau}+i(A_a-A_b)\right) \tag{3.24}$$

where there is now no prime on the second factor. These determinants can be found explicitly. We will do this shortly.

If we expand about the classical vacuum $X^a_{\text{cl}}=0=X^i_{\text{cl}}$, we find the partition function in the 1-loop approximation is

$$Z=\int dA_a\prod_{a\neq b}\frac{\det{}'\left(-d/d\tau\right)\det\left(-D_{ab}\right)\det{}^8\left(-D_{ab}+\frac{\mu}{4}\right)}{\det{}^{3/2}\left(-D_{ab}^2+\frac{\mu^2}{9}\right)\det{}^3\left(-D_{ab}^2+\frac{\mu^2}{36}\right)} \tag{3.25}$$

where $D_{ab}=\frac{d}{d\tau}-i(A_a-A_b)$. The first two terms in the numerator are the Faddeev-Popov determinant. The third term comes from fermions whereas the denominator is from bosons. Using the formula

$$\det\left(-\frac{d}{d\tau}+\omega\right)=2\sinh\frac{\beta\omega}{2}$$

with periodic boundary conditions and

$$\det\left(-\frac{d}{d\tau}+\omega\right)=2\cosh\frac{\beta\omega}{2}$$

with antiperiodic boundary conditions, we can write[2]

$$Z=\int_{-\pi}^{\pi}\prod_{a=1}^{N}\frac{d\left(\beta A_a\right)}{2\pi}\prod_{a\neq b}\frac{[1-e^{i\beta(A_a-A_b)}][1+e^{-\beta\mu/4+i\beta(A_a-A_b)}]^8}{[1-e^{-\beta\mu/3+i\beta(A_a-A_b)}]^3[1-e^{-\beta\mu/6+i\beta(A_a-A_b)}]^6} \tag{3.26}$$

[1]Using zeta-function regularization,

$$\det{}'\left(-\frac{d}{d\tau}\right)=\beta$$

.

[2]Because the matrix model action (3.5) is invariant under replacing A by A plus a constant times the unit matrix, we see that the integrand in (3.26) is indeed invariant under translating all values of A_a by the same constant.

Note that, because of supersymmetry, the zero temperature ($\beta \to \infty$) limit of the partition function is one. It also has a symmetry under replacing $e^{-\beta\mu}$ by $1/e^{-\beta\mu}$.

We must now do the remaining integral when $N \to \infty$. There are N integration variables A_a and the action, which is the logarithm of the integrand is generically of order N^2 which is large in the large N limit. For this reason, the integral can be done by saddle point integration. This amounts to finding the configuration of the variables A_a which minimize the effective action:

$$S_{\text{eff}} = \sum_{a\neq b} \big(- \ln[1 - e^{i\beta(A_a - A_b)}] - 8\ln[1 + e^{-\beta\mu/4 + i\beta(A_a - A_b)}] + \\ + 3\ln[1 - e^{-\beta\mu/3 + i\beta(A_a - A_b)}] + 6\ln[1 - e^{-\beta\mu/6 + i\beta(A_a - A_b)}]\big) \tag{3.27}$$

To study the minima, it is illuminating to Taylor expand the logarithms in the phases (this requires some assumptions of convergence for the first log)

$$S_{\text{eff}} = \sum_{n=1}^{\infty} \frac{1 - 8(-)^{n+1} r^{3n} - 3r^{4n} - 6r^{2n}}{n} \, \phi_{-n}\phi_n \tag{3.28}$$

Here,

$$r = \exp\left(-\beta\mu/12\right)$$

and

$$\phi_n = \frac{1}{N} \sum_{a=1}^{N} e^{in\beta A_a} \tag{3.29}$$

Recalling (3.19), we note that ϕ_n are multiply wound Polyakov loop operators evaluated in the static, diagonal gauge. The zeroth moment is normalized

$$\phi_0 = 1 \tag{3.30}$$

The other elements are constrained by sum rules. The density defined by

$$\begin{aligned} \rho(\chi) &= \frac{1}{N}\sum_{a=1}^{\infty} \delta(\chi - \beta A_a) \ \geq 0 \\ &= \sum_n e^{-2\pi i n\chi}\phi_n \end{aligned} \tag{3.31}$$

is a non-negative function. For example, if only ϕ_0 and $\phi_{\pm 1}$ are nonzero, (3.31) implies that $|\phi_1| \leq 1/2$.

In this one-loop approximation, the action is quadratic in the Polyakov loops. When all coefficients of the quadratic terms are positive, the action is minimized by $\phi_n = 0$ for $n \neq 0$. This is the confining phase. When a coefficient becomes negative, the effective action is minimized with one of the loops nonzero. The result is a condensation of the loops.

As we raise the temperature from zero (and lower β from infinity), the first mode to condense is $n = 1$. This occurs when

$$r_c = 1/3 \quad \to \quad T_c = \frac{\mu}{12\ln 3} \approx .0758533\mu$$

and $\phi_1 \neq 0$ when $T > T_c$.

Note that this condensation breaks a U(1) symmetry. This is associated with the center of the gauge group $U(1) \in U(N)$. It arises from the fact that all variables are in the adjoint representation. In the Euclidean path integral, gauge transformations $X(\tau) \to U(\tau)X(\tau)U^\dagger(\tau)$ must preserve the periodicity of the dynamical variables. They therefore must be periodic up to an element of the center, $U(\beta) = e^{i\theta}U(0)$. The Polyakov loop, on the other hand, being the holonomy on the time circle, does transform as $P \to e^{i\theta}P$.

Even once the static, diagonal gauge is fixed, there is a vestige of this symmetry where $\beta A_a \to \beta A_a + \theta$ or $\phi_n \to e^{in\theta}\phi_n$. This symmetry restricts the form of the effective action for Polyakov loops, so that the term with $\phi_{k_1} \ldots \phi_{k_n}$ must have $\sum k_i = 0$. It is a good symmetry of the confined phase and it is spontaneously broken in the deconfined phase. The Polyakov loop operator is an order parameter for this symmetry breaking.

3.3.3 Three-loop effective action

We now present the original work of the author of this thesis which was reported in [120]. The effective action is calculated up to three-loop order. The two and three-loop pieces of the effective action are given by the sum of connected vacuum diagrams of that loop-order. In order to calculate these, the relevant propagators must be determined.

Propagators

Our strategy here will be to construct a Euclidean Green function which obeys the equation

$$\left(-\left(\frac{d}{d\tau} - iA_{ab}\right)^2 + \omega^2\right) G_{ab}(\tau) = \delta(\tau) \tag{3.32}$$

which has the periodicity

$$G(\tau + \beta) = G(\tau) \tag{3.33}$$

We will begin by constructing $G(\tau)$ in the domain $-\beta \leq \tau \leq \beta$ and then continuing it periodically outside of this domain. For this we use the Heaviside function

$$\theta(\tau) = \begin{cases} 1 & 0 < \tau < \beta \\ 0 & -\beta < \tau < 0 \end{cases} \tag{3.34}$$

$$\theta(-\tau) = \begin{cases} 0 & 0 < \tau < \beta \\ 1 & -\beta < \tau < 0 \end{cases} \tag{3.35}$$

Then our ansätz for the Green function is

$$G(\tau) = \eta^{\tau/\beta} \left(g_+(\tau)\theta(\tau) + g_-(\tau)\theta(-\tau)\right) \tag{3.36}$$

where

$$\eta = e^{-i\beta A_{ab}} = e^{i\beta A_b}/e^{i\beta A_a} \equiv z_b/z_a \tag{3.37}$$

The Green function equation is obeyed if

$$\left(-\frac{d^2}{d\tau^2}+\omega^2\right)g_\pm(\tau)=0 \longrightarrow g_\pm(\tau)=a_\pm e^{\omega\tau}+b_\pm e^{-\omega\tau} \tag{3.38}$$

and

$$g_+(\tau)=g_-(\tau) \quad , \quad \frac{d}{d\tau}g_+(\tau)-\frac{d}{d\tau}g_-(\tau)=-1 \tag{3.39}$$

And the green function is periodic within the domain $-\beta<\tau<\beta$ if

$$\tau<0 \quad \eta g_-(\tau)=g_+(\tau+\beta) \tag{3.40}$$

The unique solution of these equations is

$$G(\tau)=\frac{\eta^{-\frac{\tau}{\beta}+1}}{2\omega}\left(\frac{e^{\omega(\tau-\beta)}}{1-\eta e^{-\omega\beta}}+\frac{e^{-\omega\tau}}{\eta-e^{-\omega\beta}}\right)\theta(\tau)+\frac{\eta^{-\frac{\tau}{\beta}}}{2\omega}\left(\frac{e^{\omega\tau}}{1-\eta e^{-\omega\beta}}+\frac{e^{-\omega(\tau+\beta)}}{\eta-e^{-\omega\beta}}\right)\theta(-\tau) \tag{3.41}$$

If needed, this Green function should be extended periodically to all values of τ.

We note that this Green function is a sum of two green functions for linear differential operators,

$$(\tau|\frac{1}{-D^2+\omega^2}|\tau')=\frac{1}{2\omega}(\tau|\frac{1}{D+\omega}+\frac{1}{-D+\omega}|\tau') \tag{3.42}$$

which implies

$$G(\tau)=\frac{1}{2\omega}\left(g_1(\tau)+g_2(\tau)\right) \tag{3.43}$$

where

$$(D+\omega)\,g_1(\tau)=\delta(\tau) \quad , \quad (-D+\omega)\,g_2(\tau)=\delta(\tau) \tag{3.44}$$

with the same periodic boundary condition that is satisfied by $G(\tau)$. We then have that

$$g_1(\tau)=\eta^{-\frac{\tau}{\beta}+1}\left(\frac{e^{\omega(\tau-\beta)}}{1-\eta e^{-\omega\beta}}\right)\theta(\tau)+\eta^{-\frac{\tau}{\beta}}\left(\frac{e^{\omega\tau}}{1-\eta e^{-\omega\beta}}\right)\theta(-\tau) \tag{3.45}$$

$$g_2(\tau)=\eta^{-\frac{\tau}{\beta}+1}\left(\frac{e^{-\omega\tau}}{\eta-e^{-\omega\beta}}\right)\theta(\tau)+\eta^{-\frac{\tau}{\beta}}\left(\frac{e^{-\omega(\tau+\beta)}}{\eta-e^{-\omega\beta}}\right)\theta(-\tau). \tag{3.46}$$

Note that $g_1^*(-\tau)=g_2(\tau)$, so that from now on we will use $g(\tau)\equiv g_1(\tau)$ only. Similarly, a fermionic propagator obeys

$$(-D+\omega)\,g_f(\tau)=\delta(\tau) \tag{3.47}$$

with the anti-periodic boundary condition

$$g_f(\tau+\beta)=-g_f(\tau) \tag{3.48}$$

We can similarly construct it in the interval $-\beta < \tau < \beta$ and continue it anti-periodically to the real line. The fermionic propagator is

$$g_f(\tau) = -\eta^{-\frac{\tau}{\beta}+1}\left(\frac{e^{-\omega\tau}}{\eta + e^{-\omega\beta}}\right)\theta(\tau) + \eta^{-\frac{\tau}{\beta}}\left(\frac{e^{-\omega(\tau+\beta)}}{\eta + e^{-\omega\beta}}\right)\theta(-\tau) \tag{3.49}$$

The full propagators are then given by

$$\left\langle X^i_{ab}(\tau)\, X^j_{cd}(\tau')\right\rangle = \frac{R}{2\omega}\,\delta^{ij}\delta_{ad}\delta_{bc}\left[g(\tau'-\tau) + g^*(\tau-\tau')\right]_{ab} \tag{3.50}$$

where, for this expression only, we can take i, j to be either flavour of scalar. For the fermions, we have

$$\begin{aligned}\left\langle(\psi_{I\alpha})_{ab}(\tau)\,(\psi^{\dagger J\beta})_{cd}(\tau')\right\rangle &= 2R\,\delta_{ad}\delta_{bc}\,\delta^\beta_\alpha\,\delta^J_I\,g_{f\,ab}(\tau'-\tau)\\ \left\langle(\psi^{\dagger I\alpha})_{ab}(\tau)\,(\psi_{J\beta})_{cd}(\tau')\right\rangle &= -2R\,\delta_{ad}\delta_{bc}\,\delta^I_J\,\delta^\alpha_\beta\,g^*_{f\,ab}(\tau-\tau')\end{aligned} \tag{3.51}$$

where $\langle\psi\psi\rangle = \langle\psi^\dagger\psi^\dagger\rangle = 0$.

2-loop diagrams

The connected vacuum diagrams at two loops may be divided into three forms, as shown in figure 3.3, where we use a dashed line to represent a fermion, while a solid line is used to indicate a scalar propagator. As an example of how these calculations are performed, we

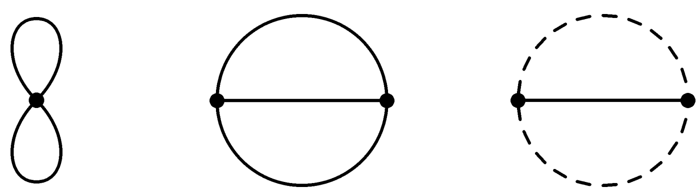


Figure 3.3: The two-loop connected vacuum diagrams for the plane-wave matrix model at finite temperature about the single five-brane vacuum. Dashed lines refer to fermions, solid lines to scalars.

will explicitly present the calculation of the third diagram in figure 3.3.

<u>Last diagram of figure 3.3</u>

The relevant term in the action is

$$\frac{1}{2}\,\mathrm{Tr}\left(+\psi^{\dagger I\alpha}\sigma^{\bar{a}\beta}_{\ \ \alpha}\,[X^{\bar{a}},\psi_{J\beta}] - \frac{1}{2}\epsilon_{\alpha\beta}\,\psi^{\dagger I\alpha}\mathrm{g}^i_{IJ}[X^i,\psi^{\dagger J\beta}] + \frac{1}{2}\epsilon^{\alpha\beta}\,\psi_{I\alpha}(\mathrm{g}^{i\dagger})^{IJ}[X^i,\psi_{J\beta}]\right). \tag{3.52}$$

The diagram comes from expanding $\exp(-S)$ to second order in the path integral. Since $\langle\psi\psi\rangle = \langle\psi^\dagger\psi^\dagger\rangle = 0$, the surviving terms are

$$\Bigg\langle \frac{1}{2}\frac{1}{4}\int d\tau\, d\tau' \Bigg\{ \mathrm{Tr}\left(\psi^\dagger\sigma^{\bar a}[X^{\bar a},\psi]\right)(\tau)\ \mathrm{Tr}\left(\psi^\dagger\sigma^{\bar b}[X^{\bar b},\psi]\right)(\tau') \\ + \mathrm{Tr}\left(-\frac{1}{2}\,\epsilon\,\psi^\dagger\,\mathsf{g}^i[X^i,\psi^\dagger]\right)(\tau)\ \mathrm{Tr}\left(\frac{1}{2}\,\epsilon\,\psi\,\mathsf{g}^j[X^j,\psi]\right)(\tau') \\ + \mathrm{Tr}\left(\frac{1}{2}\,\epsilon\,\psi\,\mathsf{g}^i[X^i,\psi]\right)(\tau)\ \mathrm{Tr}\left(-\frac{1}{2}\,\epsilon\,\psi^\dagger\,\mathsf{g}^j[X^j,\psi^\dagger]\right)(\tau')\Bigg\}\Bigg\rangle. \tag{3.53}$$

Considering the first term first, and writing it in terms of matrix indices we have

$$\frac{1}{8}\int d\tau\, d\tau'\ \left\langle \psi^\dagger_{ab}\sigma^{\bar a}\left(X^{\bar a}_{bc}\psi_{ca} - \psi_{bc}X^{\bar a}_{ca}\right)(\tau)\,\psi^\dagger_{de}\sigma^{\bar b}\left(X^{\bar b}_{ef}\psi_{fd} - \psi_{ef}X^{\bar b}_{fd}\right)(\tau')\right\rangle. \tag{3.54}$$

Keeping only the planar contributions, and noting again that $\langle\psi\psi\rangle = \langle\psi^\dagger\psi^\dagger\rangle = 0$ this becomes

$$\frac{1}{8}\int d\tau\, d\tau'\,\sigma^{\bar a}\sigma^{\bar b}\left\{\left\langle X^{\bar a}_{ca}X^{\bar b}_{fd}\right\rangle\left\langle\psi_{bc}\psi^\dagger_{de}\right\rangle\left\langle\psi^\dagger_{ab}\psi_{ef}\right\rangle + \left\langle X^{\bar a}_{bc}X^{\bar b}_{ef}\right\rangle\left\langle\psi_{ca}\psi^\dagger_{de}\right\rangle\left\langle\psi^\dagger_{ab}\psi_{fd}\right\rangle\right\} \tag{3.55}$$

where the first field in each expectation value is evaluated at τ and the second at τ'. Recalling the form of the propagators (3.50) and (3.51) we have:

$$-8\frac{(2R)^2}{8}\sum_{abc}\int d\tau\, d\tau'\ \left[\frac{3R}{2\omega_1}(g+g^*_-)^{\omega_1}_{ca}\right]\left[(g_f)_{bc}(g^*_{f-})_{ab} + (g_f)_{ab}(g^*_{f-})_{bc}\right] \tag{3.56}$$

where the subscript "$-$" indicates time reversal. The factor of 8 comes from $\delta^I_I\,\mathrm{Tr}\,\sigma\sigma$. The factor of 3 from the fact that there are three scalars of the first flavour. Now, noting that $\omega_1 = \mu/3$ we have

$$-\frac{2R^3}{\mu}\sum_{abc}\int d\tau\, d\tau'\ \left[9\,(g+g^*_-)^{\omega_1}_{ca}\right]\left[(g_f)_{bc}(g^*_{f-})_{ab} + (g_f)_{ab}(g^*_{f-})_{bc}\right]. \tag{3.57}$$

Now we attack the fermion propagator terms

$$(g_f)_{bc}(g^*_{f-})_{ab} = \phi_{bc}^{(\tau'-\tau)}\left[\frac{-\phi_{bc}^{-\beta}}{1+\phi_{bc}^{-\beta}}\theta + \frac{1}{1+\phi_{bc}^{-\beta}}\bar\theta\right]\phi_{ab}^{*(\tau-\tau')}\left[\frac{-\phi_{ab}^{*-\beta}}{1+\phi_{ab}^{*-\beta}}\bar\theta + \frac{1}{1+\phi_{ab}^{*-\beta}}\theta\right] \tag{3.58}$$

where $\phi_{ab} = e^{iA_{ab}+\omega}$, $\theta = \theta(\tau'-\tau)$, and $\bar\theta \equiv \theta(\tau-\tau')$. Using the fact that $\theta^2 = \theta$ and that $\theta\bar\theta = 0$, and that $A_{ab} \equiv A_a - A_b$, we have:

$$(g_f)_{bc}(g^*_{f-})_{ab} = -e^{iA_{ac}(\tau'-\tau)}\frac{\phi_{bc}^{-\beta}\theta + \phi_{ab}^{*-\beta}\bar\theta}{(1+\phi_{bc}^{-\beta})(1+\phi_{ab}^{*-\beta})}. \tag{3.59}$$

Therefore

$$(g_f)_{bc}(g^*_{f-})_{ab} + (g_f)_{ab}(g^*_{f-})_{bc} = \\ -e^{iA_{ac}(\tau'-\tau)} \left\{ \left[\frac{\phi_{bc}^{-\beta}}{(1+\phi_{bc}^{-\beta})(1+\phi_{ab}^{*-\beta})} + \frac{\phi_{ab}^{-\beta}}{(1+\phi_{ab}^{-\beta})(1+\phi_{bc}^{*-\beta})} \right] \theta + \left[\text{c.c.} \right] \bar{\theta} \right\} \tag{3.60}$$

while for the scalar propagators we have

$$\left[g + g^*_-\right]^\omega_{ca} = \frac{1}{|1-\phi_{ca}^{-\beta}|^2} \left\{ \left[\mathcal{A}_{ca}\phi_{ca}^{(\tau'-\tau)} + \mathcal{B}_{ca}\phi_{ca}^{*(\tau-\tau')} \right] \theta + \left[\mathcal{A}^*_{ca}\phi_{ca}^{*(\tau-\tau')} + \mathcal{B}^*_{ca}\phi_{ca}^{(\tau'-\tau)} \right] \bar{\theta} \right\} \tag{3.61}$$

where,

$$\mathcal{A}_{ca} = (1-\phi_{ca}^{*-\beta})\phi_{ca}^{-\beta} \qquad \mathcal{B}_{ca} = (1-\phi_{ca}^{-\beta}). \tag{3.62}$$

Now it can be seen that,

$$\left[(g_f)_{bc}(g^*_{f-})_{ab} + (g_f)_{ab}(g^*_{f-})_{bc}\right] \left[g + g^*_-\right]^\omega_{ca} = G\theta + H\bar{\theta} \tag{3.63}$$

and by changing variables in the second term such that τ and τ' are interchanged, one notes that $H \to G^*$, so that

$$\left[(g_f)_{bc}(g^*_{f-})_{ab} + (g_f)_{ab}(g^*_{f-})_{bc}\right] \left[g + g^*_-\right]^\omega_{ca} = (G + G^*)\,\theta = 2\mathrm{Re}(G)\,\theta. \tag{3.64}$$

Now notice that the $e^{iA_{ac}(\tau'-\tau)}$ term from the fermion propagators kills the gauge field dependence of the scalar propagators

$$e^{iA_{ac}(\tau'-\tau)}\phi_{ca}^{(\tau'-\tau)} = e^{\omega(\tau'-\tau)} \qquad e^{iA_{ac}(\tau'-\tau)}\phi_{ca}^{*(\tau-\tau')} = e^{-\omega(\tau'-\tau)}. \tag{3.65}$$

Thus yielding the following form for G

$$G = -\frac{\left(\mathcal{A}_{ca}e^{\omega(\tau'-\tau)} + \mathcal{B}_{ca}e^{-\omega(\tau'-\tau)}\right)\left(\bar{\mathcal{A}}_{bc}\bar{\mathcal{B}}_{ab} + \bar{\mathcal{A}}_{ab}\bar{\mathcal{B}}_{bc}\right)}{C_{ca}\bar{C}_{ab}\bar{C}_{bc}} \tag{3.66}$$

where

$$\begin{aligned} &C_{ab} = |1-\phi_{ab}^{-\beta}(\omega_i)|^2 \qquad \bar{C}_{ab} = |1+\phi_{ab}^{-\beta}(\omega_f)|^2 \\ &\mathcal{A}_{ab} = \left[1-\phi_{ab}^{*-\beta}(\omega_i)\right]\phi_{ab}^{-\beta}(\omega_i) \qquad \bar{\mathcal{A}}_{ab} = \left[1+\phi_{ab}^{*-\beta}(\omega_f)\right]\phi_{ab}^{-\beta}(\omega_f) \\ &\mathcal{B}_{ab} = 1-\phi_{ab}^{-\beta}(\omega_i) \qquad \bar{\mathcal{B}}_{ab} = 1+\phi_{ab}^{-\beta}(\omega_f). \end{aligned} \tag{3.67}$$

The integrations over τ and τ' are performed using

$$\int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau'\, \theta(\tau'-\tau)\, e^{\omega(\tau'-\tau)} = \frac{e^{\omega\beta} - 1 - \beta\omega}{\omega^2}. \tag{3.68}$$

The contribution (3.54) may then be reduced to the following form

$$\begin{aligned}\frac{-2}{C^{\omega_1}_{ca}\bar{C}_{ab}\bar{C}_{bc}}\frac{3\beta}{\mu}\Big\{&\pm\left[\cos\beta A_{ab}+\cos\beta A_{bc}\right]\\ &\times\left[e^{-\beta\mu/4}+e^{-3\beta\mu/4}+e^{-11\beta\mu/12}+e^{-17\beta\mu/12}-2e^{-7\beta\mu/12}-2e^{-13\beta\mu/12}\right]\\ &+\left[2+2\cos\beta A_{ca}\right]\left[e^{-\beta\mu/2}+e^{-7\beta\mu/6}-2e^{-5\beta\mu/6}\right]\Big\}\left(-9\,\frac{2R^3}{\mu}\right)\end{aligned} \tag{3.69}$$

where the negative sign comes from (3.66), the factor of 2 from (3.64), the factor of $3\beta/\mu$ from the final reduction, and the final factor of $-2\cdot 9R^3/\mu$ from (3.57).
The form of the C's is

$$C_{ab}=1-2e^{-\beta\omega}\cos\beta A_{ab}+e^{-2\beta\omega}\qquad \bar{C}_{ab}=1+2e^{-\beta\omega}\cos\beta A_{ab}+e^{-2\beta\omega}. \tag{3.70}$$

Now for the second and third terms in (3.53), starting with the second term

$$-\frac{1}{32}\int d\tau\,d\tau'\,\left\langle \epsilon\,\psi^\dagger_{ab}\,\mathsf{g}^i\left(X^i_{bc}\psi^\dagger_{ca}-\psi^\dagger_{bc}X^i_{ca}\right)(\tau)\;\epsilon\,\psi_{de}\,\mathsf{g}^{\dagger j}\left(X^j_{ef}\psi_{fd}-\psi_{ef}X^j_{fd}\right)(\tau')\right\rangle. \tag{3.71}$$

There are more planar contributions here than for the first term of (3.53), we have

$$\begin{aligned}-\frac{1}{32}\int d\tau\,d\tau'\;\epsilon\,\epsilon\,\mathsf{g}^i\mathsf{g}^{\dagger j}\Big\{&\langle X^i_{ca}X^j_{fd}\rangle\left\langle\psi^\dagger_{bc}\psi_{de}\right\rangle\left\langle\psi^\dagger_{ab}\psi_{ef}\right\rangle+\langle X^i_{bc}X^j_{ef}\rangle\langle\psi^\dagger_{ca}\psi_{de}\rangle\left\langle\psi^\dagger_{ab}\psi_{fd}\right\rangle\\ &+\langle X^i_{ca}X^j_{ef}\rangle\left\langle\psi^\dagger_{bc}\psi_{fd}\right\rangle\left\langle\psi^\dagger_{ab}\psi_{de}\right\rangle+\langle X^i_{bc}X^j_{fd}\rangle\langle\psi^\dagger_{ca}\psi_{ef}\rangle\left\langle\psi^\dagger_{ab}\psi_{de}\right\rangle.\Big\}\end{aligned} \tag{3.72}$$

Not surprisingly each term contributes the same quantity, and a factor of four is gained, the result is

$$+\frac{8\cdot 4}{32}(2R)^2\cdot 6\sum_{abc}\int d\tau\,d\tau'\,\left[\frac{R}{2\omega_2}(g+g^*_-)^{\omega_2}_{ca}\right]\left[(g^*_{f-})_{ab}(g^*_{f-})_{bc}\right]. \tag{3.73}$$

Here we encounter the structure $\mathrm{Tr}\,\epsilon^2\,\mathrm{Tr}\,\mathsf{g}^i\mathsf{g}^{\dagger i}=-8\cdot 6$, where the sign comes from the fact that $\epsilon=i\sigma^2$. The factor of 6 counts the six scalars of the second flavour. The third term in (3.53) is identical except that $\langle\psi^\dagger\psi\rangle\to\langle\psi\psi^\dagger\rangle$, and thus the full expression is

$$\frac{8\cdot 4}{32}(2R)^2\cdot 6\sum_{abc}\int d\tau\,d\tau'\,\left[\frac{R}{2\omega_2}(g+g^*_-)^{\omega_2}_{ca}\right]\left[(g^*_{f-})_{ab}(g^*_{f-})_{bc}+(g_f)_{ab}(g_f)_{bc}\right]. \tag{3.74}$$

or more concisely,

$$\frac{2R^3}{\mu}\sum_{abc}\int d\tau\, d\tau' \, \left[36\,(g+g_-^*)_{ca}^{\omega_2}\right]\left[(g_{f-}^*)_{ab}(g_{f-}^*)_{bc}+(g_f)_{ab}(g_f)_{bc}\right]. \tag{3.75}$$

We continue as before by evaluating the fermion propagators

$$\begin{aligned}(g_f)_{ab}(g_f)_{bc} &= e^{(iA_{ac}+2\omega_f)(\tau'-\tau)}\frac{\phi_{ab}^{-\beta}\phi_{bc}^{-\beta}\theta+\bar\theta}{(1+\phi_{bc}^{-\beta})(1+\phi_{ab}^{-\beta})}\\ (g_{f-}^*)_{ab}(g_{f-}^*)_{bc} &= e^{(iA_{ac}-2\omega_f)(\tau'-\tau)}\frac{\phi_{ab}^{*-\beta}\phi_{bc}^{*-\beta}\bar\theta+\theta}{(1+\phi_{bc}^{*-\beta})(1+\phi_{ab}^{*-\beta})}.\end{aligned} \tag{3.76}$$

Now the same integration variable switch employed in (3.63) and (3.64) can be used here, yielding

$$G=\frac{\left(\mathcal{A}_{ca}e^{\omega_2(\tau'-\tau)}+\mathcal{B}_{ca}e^{-\omega_2(\tau'-\tau)}\right)\left(\bar{\mathcal{A}}_{ab}\bar{\mathcal{A}}_{bc}e^{2\omega_f(\tau'-\tau)}+\bar{\mathcal{B}}_{ab}\bar{\mathcal{B}}_{bc}e^{-2\omega_f(\tau'-\tau)}\right)}{C_{ca}\bar C_{ab}\bar C_{bc}}. \tag{3.77}$$

One may then reduce to obtain the final result

$$\begin{aligned}\frac{2}{C_{ca}^{\omega_2}\bar C_{ab}\bar C_{bc}}\frac{3\beta}{2\mu}\Bigg\{&\pm\left[\cos\beta A_{ab}+\cos\beta A_{bc}\right]\\ &\times\left[e^{-\beta\mu/4}+e^{-5\beta\mu/12}+e^{-11\beta\mu/12}+e^{-13\beta\mu/12}-2e^{-3\beta\mu/4}-2e^{-7\beta\mu/12}\right]\\ &+\cos\beta A_{ca}\left[e^{-7\beta\mu/6}+e^{-\beta\mu/6}-e^{-\beta\mu/2}-e^{-5\beta\mu/6}\right]\\ &+1+e^{-4\beta\mu/3}+2e^{-2\beta\mu/3}-2e^{-\beta\mu/3}-2e^{-\beta\mu}\Bigg\}\left(36\,\frac{2R^3}{\mu}\right)\end{aligned} \tag{3.78}$$

where the factor of 2 comes from (3.64), the factor of $3\beta/2\mu$ from the final reduction, and the final factor of $2\cdot 36R^3/\mu$ from (3.75).

Final 2-loop effective action result

The other diagrams in figure 3.3 are similarly calculated. The details are presented in appendix E. The final result may be stated in a compact form using the variable $r\equiv\exp(-\beta\mu/12)$

$$
\begin{aligned}
S_{\text{eff}}^{\text{2-loops}} &= \frac{27\beta R^3}{4\mu^2}\Bigg(-\frac{(1-r^8)}{C_{ab}^{\omega_1}C_{ca}^{\omega_1}} - 20\frac{(1-r^4)}{C_{ab}^{\omega_2}C_{ca}^{\omega_2}} - 12\frac{(1-r^8)(1-r^4)}{C_{ab}^{\omega_1}C_{ca}^{\omega_2}} \\
&+\frac{(r^8+4r^4+1)(r^4-1)^4+[\cos\beta A_{ab}+\cos\beta A_{bc}+\cos\beta A_{ca}]\,2r^4(r^4-1)^4}{C_{ab}^{\omega_1}C_{bc}^{\omega_1}C_{ca}^{\omega_1}} \\
&+16\frac{r^3(r^4-r^2+1)(r^4-1)^2(r^2+1)\left[\cos\beta A_{ab}+\cos\beta A_{bc}\right]+r^6(r^4-1)^2\left[2+2\cos\beta A_{ca}\right]}{\bar{C}_{ab}\bar{C}_{bc}C_{ca}^{\omega_1}} \\
&+32\frac{r^3(r^4-1)^2(r^2+1)\left[\cos\beta A_{ab}+\cos\beta A_{bc}\right]+r^2(r^8-1)(r^4-1)\cos\beta A_{ca}+(r^4-1)^2(r^8+1)}{\bar{C}_{ab}\bar{C}_{bc}C_{ca}^{\omega_2}}\Bigg)
\end{aligned}
\tag{3.79}
$$

where we have

$$
\begin{aligned}
C_{ab}^{\omega_1} &= 1-2\,r^4\,\cos\beta A_{ab}+r^8 \\
C_{ab}^{\omega_2} &= 1-2\,r^2\,\cos\beta A_{ab}+r^4 \\
\bar{C}_{ab} &= 1+2\,r^3\,\cos\beta A_{ab}+r^6.
\end{aligned}
\tag{3.80}
$$

In the zero temperature limit, $r\to 0$ and $C,\bar{C}\to 1$. We then see that the free energy is $1+20+12-1-32=0$, and so there is a SUSY cancellation at zero temperature. In order restate this result in terms of the multiply-wound Polyakov loops, we use the following identities

$$
\begin{aligned}
(C_{ab}^{\omega_1})^{-1} &= \sum_n \frac{r^{4|n|}}{1-r^8}\eta_a^{-n}\eta_b^n \\
(C_{ab}^{\omega_2})^{-1} &= \sum_n \frac{r^{2|n|}}{1-r^4}\eta_a^{-n}\eta_b^n \\
\left(\bar{C}_{ab}\right)^{-1} &= \sum_n \frac{r^{3|n|}\,(-1)^n}{1-r^6}\eta_a^{-n}\eta_b^n
\end{aligned}
\tag{3.81}
$$

where $\eta_a=\exp(i\beta A_a)$. So that, for example, we may express quantities such as the following

$$
\begin{aligned}
\left(\bar{C}_{ab}\bar{C}_{ab}C_{ab}^{\omega_2}\right)^{-1} &= \sum_{\substack{nmp\\abc}} \frac{(-1)^n r^{3|n|}}{1-r^6}\frac{(-1)^m r^{3|m|}}{1-r^6}\frac{r^{2|p|}}{1-r^4}\eta_a^{-n}\eta_b^n\eta_b^{-m}\eta_c^m\eta_c^{-p}\eta_a^p \\
&= \frac{1}{(1-r^6)^2}\frac{1}{1-r^4}\sum_{pn}\sum_{abc}\eta_a^p\eta_b^n\eta_c^{-p-n}(-1)^n\sum_m r^{(3|m|+3|m+n|+2|m+n+p|)} \\
&= \frac{N^3}{(1-r^6)^2}\frac{1}{1-r^4}\sum_{pn}\phi_p\phi_n\phi_{-p-n}(-1)^n\sum_m r^{(3|m|+3|m+n|+2|m+n+p|)}
\end{aligned}
\tag{3.82}
$$

where the sum over m is a straightforward, if tedious, application of geometric series, and can be performed analytically. We may use this method to re-express (3.79) as

$$\begin{aligned}\frac{1}{N^2}\,S_{\text{eff}}^{\text{2-loops}} &= 3\lambda\ln(r)\sum_{mn}\phi_n\,\phi_m\,\phi_{-m-n}\Bigg[-4\frac{r^8-r^4+1}{(r^4+1)^2}r^{4|n|+4|m|}\\ &+16\frac{(r^4-r^2+1)(r^2+1)^2}{(r^4+r^2+1)(r^4+1)}(-1)^n\,r^{3|n|+4|m|}-16\frac{(r^2+1)^2}{r^4+r^2+1}(-1)^{n+m}\,r^{3|n|+3|m|}\\ &+32\frac{(r^2+1)^2}{r^4+r^2+1}(-1)^n\,r^{3|n|+2|m|}-12\,r^{4|n|+2|m|}-20\,r^{2|n|+2|m|}\\ &-4\frac{(r^4-1)(r^8+r^4+1)}{(r^4+1)^3}F_{mn}(4,4)+16\frac{(r^2+1)(r^{10}-1)}{(r^4+1)(r^4+r^2+1)^2}(-1)^m\,F_{mn}(3,4)\\ &-16\frac{(r^2+1)(r^6-r^4+r^2-1)}{(r^4+r^2+1)^2}(-1)^m\,F_{mn}(3,2)\Bigg]\end{aligned} \tag{3.83}$$

where we have used the coupling $\lambda = (3R/\mu)^3 N$, and define the function $F_{mn}(a,b)$ in the following manner

$$F_{mn}(a,b) = \begin{cases} F^1_{mn}(a,b) & m,n\geq 0 \quad \text{or} \quad m,n<0 \\ F^2_{mn}(a,b) & n<0, m\geq -n \quad \text{or} \quad n\geq 0, m<-n \\ F^3_{mn}(a,b) & m<0, m\geq -n \quad \text{or} \quad m\geq 0, m<-n \end{cases} \tag{3.84}$$

where

$$\begin{aligned}F^1_{mn}(a,b) = r^{a(2+n+m)+b}\Bigg[&\frac{r^{b(n+m)-an}}{1-r^{2a+b}}+\frac{r^{-b-2a+an}}{1-r^{2a+b}}+\frac{r^{-2a-an+b(n+m)}}{r^b-1}\\ &-\frac{r^{-2a-an+bn}}{r^b-1}-\frac{r^{-2a-an+bn}}{-r^b+r^{2a}}+\frac{r^{-2a+an}}{-r^b+r^{2a}}\Bigg]\end{aligned} \tag{3.85}$$

$$\begin{aligned}F^2_{mn}(a,b) = r^{a(n+m)+b}\Bigg[&\frac{r^{2a-an+b(n+m)}}{1-r^{2a+b}}+\frac{r^{-b-an-bn}}{1-r^{2a+b}}\\ &+\frac{r^{-an+b(n+m)}+r^{-b-an}-r^{-an}}{r^b-1}-\frac{r^{-bn-b-an}}{1-r^b}\Bigg]\end{aligned} \tag{3.86}$$

$$\begin{aligned}F^3_{mn}(a,b) = r^{a(n-m)+b}\Bigg[&\frac{r^{-an+2a+bn}}{1-r^{2a+b}}+\frac{r^{-b+an+2am}}{1-r^{2a+b}}+\frac{r^{-an+bn}}{r^b-1}\\ &-\frac{r^{-an+b(n+m)}}{r^b-1}-\frac{r^{-an+b(n+m)}}{-r^b+r^{2a}}+\frac{r^{an+2am}}{-r^b+r^{2a}}\Bigg].\end{aligned} \tag{3.87}$$

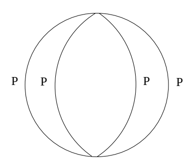
3-loop diagrams

As one can imagine, the complexity at the three-loop level is far greater than at 2-loops. An extensive C++ code was written by the author to produce all three loop diagrams, and their associated combinatoric prefactors. The output of this code may be summarized as follows. We introduce some new notation to simplify the presentation

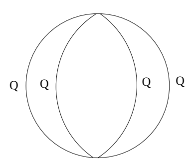
$$
\begin{aligned}
P_{ab}(t_{21}) &= \frac{R}{2\omega_1}[g(t_2-t_1)+g^*(t_1-t_2)]_{ab}^{\omega_1} \\
Q_{ab}(t_{21}) &= \frac{R}{2\omega_2}[g(t_2-t_1)+g^*(t_1-t_2)]_{ab}^{\omega_2} \\
F_{ab}(t_{21}) &= 2R\, g_{f\,ab}(t_2-t_1) \\
G_{ab}(t_{21}) &= -2R\, g^*_{f\,ab}(t_1-t_2).
\end{aligned}
\tag{3.88}
$$

Cat's eye diagram

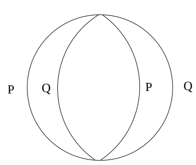
Results should be multiplied by R^2.



$$3\,\frac{3}{2}P_{ab}(t_{10})P_{bc}(t_{10})P_{cd}(t_{10})P_{da}(t_{10})$$



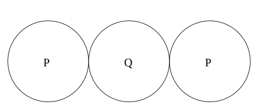
$$15\,\frac{3}{2}Q_{ab}(t_{10})Q_{bc}(t_{10})Q_{cd}(t_{10})Q_{da}(t_{10})$$



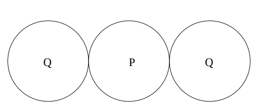
$$9\,P_{ab}(t_{10})P_{bc}(t_{10})Q_{cd}(t_{10})Q_{da}(t_{10}) + 18\,P_{ab}(t_{10})Q_{bc}(t_{10})P_{cd}(t_{10})Q_{da}(t_{10})$$

Triple bubble diagram

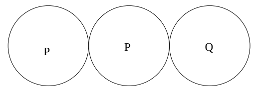
Results should be multiplied by R^2.



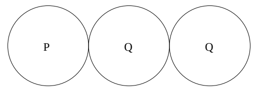
$$27\Big[P_{ab}(0)P_{cd}(0)+P_{ab}(0)P_{ad}(0)\Big]Q_{ac}(t_{10})Q_{ca}(t_{10})$$



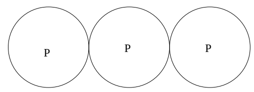
$$54\Big[Q_{ab}(0)Q_{cd}(0)+Q_{ab}(0)Q_{ad}(0)\Big]P_{ac}(t_{10})P_{ca}(t_{10})$$



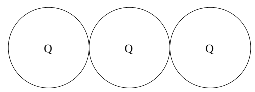
$$36\Big[P_{ab}(0)Q_{cd}(0)+P_{ab}(0)Q_{ad}(0)\Big]P_{ac}(t_{10})P_{ca}(t_{10})$$



$$90\Big[P_{ab}(0)Q_{cd}(0)+P_{ab}(0)Q_{ad}(0)\Big]Q_{ac}(t_{10})Q_{ca}(t_{10})$$



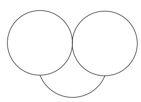
$$12\,P_{ab}(0)P_{cd}(0)P_{ac}(t_{10})P_{ca}(t_{10})$$



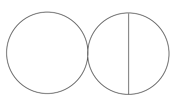
$$150\,Q_{ab}(0)Q_{cd}(0)Q_{ac}(t_{10})Q_{ca}(t_{10})$$

Theta-bubble diagram

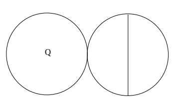
Results should be multiplied by R.



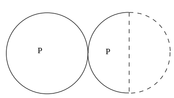
$$-\,9\,P_{ab}(t_{10})P_{bc}(t_{10})P_{ac}(t_{21})P_{cd}(t_{20})P_{da}(t_{20})$$



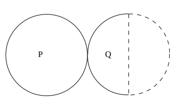
$$-\,12\,P_{ab}(0)P_{ca}(t_{10})P_{ac}(t_{20})P_{cd}(t_{21})P_{da}(t_{21})$$



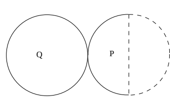
$$-\,36\,Q_{ab}(0)P_{ca}(t_{10})P_{ac}(t_{20})P_{cd}(t_{21})P_{da}(t_{21})$$



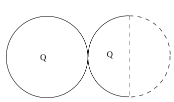
$$-\,12\,P_{ab}(0)P_{ca}(t_{10})P_{ac}(t_{20})\Big[F_{cd}(t_{21})G_{da}(t_{21})+F\leftrightarrow G\Big]$$



$$-\,36\,P_{ab}(0)Q_{ca}(t_{10})Q_{ac}(t_{20})\Big[F_{cd}(t_{21})F_{da}(t_{21})+F\leftrightarrow G\Big]$$

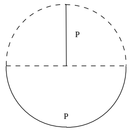


$$-\,36\,Q_{ab}(0)P_{ca}(t_{10})P_{ac}(t_{20})\Big[F_{cd}(t_{21})G_{da}(t_{21})+F\leftrightarrow G\Big]$$

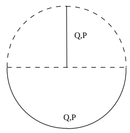


$$-\,60\,Q_{ab}(0)Q_{ca}(t_{10})Q_{ac}(t_{20})\Big[G_{cd}(t_{21})G_{da}(t_{21})+F\leftrightarrow G\Big]$$

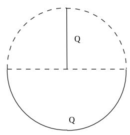
Circle-T Diagram



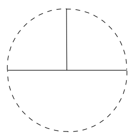
$$-\,\frac{3}{8}\,P_{ab}(t_{20})P_{cd}(t_{31})\Big[F_{bd}(t_{10})F_{bc}(t_{21})F_{ac}(t_{32})G_{da}(t_{30})+F\leftrightarrow G\Big]$$



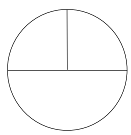
$$-\frac{9}{2}\,P_{ab}(t_{20})Q_{cd}(t_{31})\Big[F_{bd}(t_{10})G_{bc}(t_{21})G_{ac}(t_{32})G_{da}(t_{30}) + F \leftrightarrow G\Big]$$



$$-\,3\,Q_{ab}(t_{20})Q_{cd}(t_{31})\Big[G_{bd}(t_{10})F_{bc}(t_{21})G_{ac}(t_{32})G_{da}(t_{30}) + F \leftrightarrow G\Big]$$

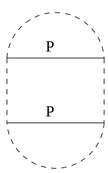


$$-\,2\,P_{ab}(t_{30})P_{ca}(t_{31})P_{bc}(t_{32})\Big[F_{bd}(t_{20})G_{dc}(t_{21})G_{da}(t_{10}) - F \leftrightarrow G\Big]$$

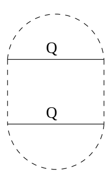


$$\frac{1}{2}\,P_{ab}(t_{10})P_{bc}(t_{20})P_{ca}(t_{30})P_{db}(t_{21})P_{ad}(t_{31})P_{dc}(t_{32})$$

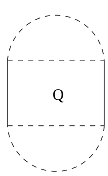
<u>Two-rung ladder diagrams</u>



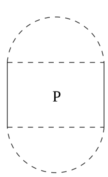
$$\frac{9}{4}\,P_{ab}(t_{10})\Big[P_{cd}(t_{32})G_{bc}(t_{32}) + P_{bc}(t_{32})G_{cd}(t_{32})\Big]G_{bd}(t_{20})G_{bd}(t_{13})F_{da}(t_{10}) + (F \leftrightarrow G)$$



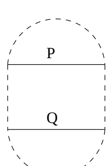
$$9\,Q_{ab}(t_{10})\Big[Q_{cd}(t_{32})G_{bc}(t_{32}) + Q_{bc}(t_{32})G_{cd}(t_{32})\Big]F_{bd}(t_{20})F_{bd}(t_{13})F_{da}(t_{10}) + (F \leftrightarrow G)$$



$$6\,Q_{ab}(t_{10})Q_{ba}(t_{32})F_{bc}(t_{20})F_{ca}(t_{20})\Big[F_{ad}(t_{31})F_{db}(t_{31}) + G_{db}(t_{31})G_{ad}(t_{31})\Big] + (F \leftrightarrow G)$$

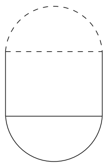


$$3\,P_{ab}(t_{10})P_{ba}(t_{32})F_{bc}(t_{20})G_{ca}(t_{20})\Big[F_{ad}(t_{31})G_{db}(t_{31}) + F_{db}(t_{31})G_{ad}(t_{31})\Big] + (F \leftrightarrow G)$$

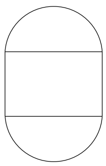


$$-\,9\,P_{ab}(t_{10})Q_{cd}(t_{32})\Big[F_{bd}(t_{20})G_{da}(t_{10})G_{db}(t_{31})G_{bc}(t_{32}) + (F \leftrightarrow G)\Big]$$

$$-9\, P_{ab}(t_{10})Q_{ca}(t_{32})\Big[F_{da}(t_{20})G_{bd}(t_{10})G_{ad}(t_{31})G_{dc}(t_{32}) + (F \leftrightarrow G)\Big]$$



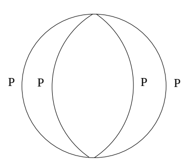
$$6\, P_{ab}(t_{20})P_{ba}(t_{31})P_{ac}(t_{32})P_{cb}(t_{32})\Big[F_{bd}(t_{10})G_{da}(t_{10}) + (F \leftrightarrow G)\Big]$$



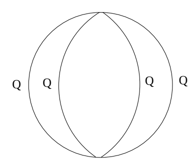
$$3\, P_{ab}(t_{10})P_{ca}(t_{10})P_{bc}(t_{20})P_{cb}(t_{31})P_{bd}(t_{32})P_{dc}(t_{32})$$

Processing the 3-loop diagrams

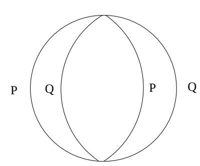
The diagrams must be processed and integrated as is done in appendix E for the 2-loop diagrams. Obviously, this process is horribly complicated and was achieved via computer algebra systems. The output of each diagram was obtained in a form analogous to (3.79). It would be far too cumbersome to display those results here. The zero temperature limit was then taken in order to verify the SUSY cancellation. The results of this zero temperature limit are as follows.



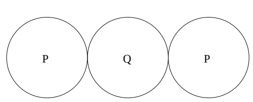
$$= \frac{2187}{64}\frac{\beta R^6}{\mu^5}$$



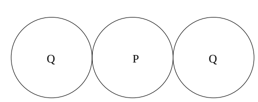
$$= \frac{10935}{2}\frac{\beta R^6}{\mu^5}$$



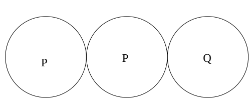
$$= \frac{2187}{2}\frac{\beta R^6}{\mu^5}$$



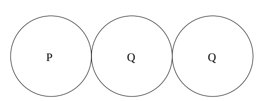
$$= 6561\,\frac{\beta R^6}{\mu^5}$$



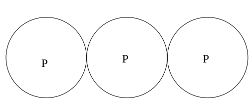
$$= 6561\,\frac{\beta R^6}{\mu^5}$$



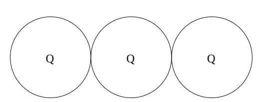
$$= 2187\,\frac{\beta R^6}{\mu^5}$$



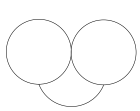
$$= 43740\,\frac{\beta R^6}{\mu^5}$$



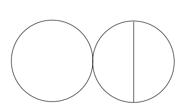
$$= \frac{729}{4}\frac{\beta R^6}{\mu^5}$$



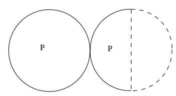
$$= 72900\,\frac{\beta R^6}{\mu^5}$$



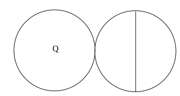
$$= -\frac{10935}{32}\frac{\beta R^6}{\mu^5}$$



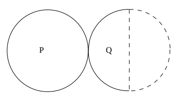
$$= -729\,\frac{\beta R^6}{\mu^5}$$



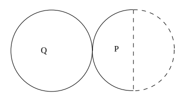
$$= 0$$



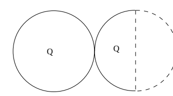
$$= -4374\,\frac{\beta R^6}{\mu^5}$$



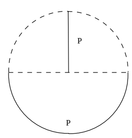
$$= -43740\,\frac{\beta R^6}{\mu^5}$$



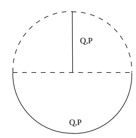
$$= 0$$



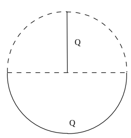
$$= -145800\,\frac{\beta R^6}{\mu^5}$$



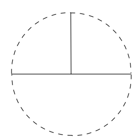
$$= 0$$



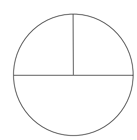
$$= 2916\,\frac{\beta R^6}{\mu^5}$$



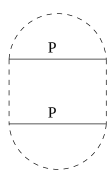
$$= 7776\,\frac{\beta R^6}{\mu^5}$$



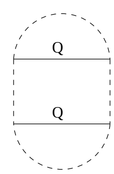
$$= 0$$



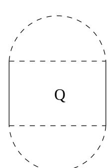
$$= \frac{6561}{64}\frac{\beta R^6}{\mu^5}$$



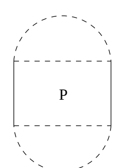
$$= 0$$



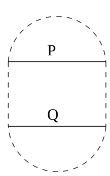
$$= -34992\,\frac{\beta R^6}{\mu^5}$$



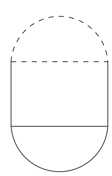
$$= 73872\,\frac{\beta R^6}{\mu^5}$$



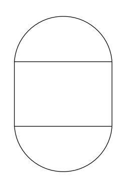
$$= 0$$



$$= 5832\,\frac{\beta R^6}{\mu^5}$$



$$= 0$$



$$= \frac{24057}{32}\frac{\beta R^6}{\mu^5}$$

One can check that the sum of the above factors is zero, as guaranteed by SUSY. The next step is to extract the $|\phi_1|^4$ terms in the expression analogous to (3.83). These $\mathcal{O}(\lambda^2)$ terms are then combined with the one-loop ($\mathcal{O}(\lambda^0)$) and two-loop ($\mathcal{O}(\lambda)$) results, in order to assemble

$$\frac{1}{N^2}S_{\text{eff}} = \Delta_1(r)|\phi_1|^2 + \Delta_2(r)|\phi_2|^2 + \ \lambda\ P_1(r)\left(\phi_1\phi_1\phi_{-2} + \text{c.c.}\right) + \lambda^2\ P_2(r)\,|\phi_1|^4 + \ldots \tag{3.89}$$

The coefficients in (3.89) are found to be

$$\begin{aligned}\Delta_1(r) = &\left[1-8r^3-3r^4-6r^2\right] - 24\lambda\ \left[\ln(r)\, r^2(r^2+1)(r+1)^4\right] - \\ &-3\lambda^2 r^2\ \Big[\ \ln(r)^2\ \big(68\, r^{10}+352\, r^9+904\, r^8+1536\, r^7+2256\, r^6+ \\ &\qquad +3104\, r^5+4120\, r^4+2304\, r^3+928\, r^2+192\, r+16\big) - \\ &\quad -\ln(r)\ \big(27\, r^{10}+152\, r^9+390\, r^8+640\, r^7+915\, r^6+1232\, r^5+ \\ &\qquad +1748\, r^4+1184\, r^3+466\, r^2+440\, r+102\big)\ \Big] + ... \end{aligned} \tag{3.90}$$

$$\Delta_2(r) = \frac{1}{2}\left(1+8r^6-3r^8-6r^4\right) + \dots \tag{3.91}$$

$$P_1(r) = -12\ln(r)\, r^4(2r^4-4r^3+3r^2-4r+5)(r+1)^4 \tag{3.92}$$

$$\begin{aligned}P_2(r) = 3r^4\Big[&-\ln(r)^2\ \big(\ 136\, r^{12}+512\, r^{11}+704\, r^{10}-1308\, r^8-1376\, r^7+1560\, r^6 \\ &\qquad +6400\, r^5+10896\, r^4+8096\, r^3+2136\, r^2+1536\, r+240\ \big) + \\ &+\ln(r)\ \big(\ 27\, r^{12}+120\, r^{11}+166\, r^{10}-32\, r^9-271\, r^8-16\, r^7+1044\, r^6+ \\ &\qquad +2624\, r^5+4036\, r^4+3256\, r^3+774\, r^2+768\, r+944\ \big)\ \Big] + \dots \end{aligned} \tag{3.93}$$

Eliminating ϕ_2 using its equation of motion, we obtain the effective action for ϕ_1, in the large N limit, and to order λ^2

$$\frac{1}{N^2}S_{\text{eff}} = \Delta_1(r)|\phi_1|^2 + \lambda^2\left(P_2(r) - \frac{\left[P_1(r)\right]^2}{\Delta_2(r)}\right)|\phi_1|^4 + \dots \tag{3.94}$$

3.3.4 Phase transition

Inspecting (3.89) and (3.94) we see why it was necessary to compute the effective action to three-loop order in order to discover the first correction to the 1-loop transition temperature $r_c = 1/3$. It is because the next highest term in the action for the lowest Polyakov loop ϕ_1 is quartic, and its coefficient at leading order receives contributions both from the 2-loop order squared, i.e. $(\lambda P_1(r))^2$, and the 3-loop order, i.e. $\lambda^2 P_2(r)$. One can check that this quartic coefficient is negative over the entire range of $r \in [0,1]$. We therefore have the following picture of the effective potential for ϕ_1. At low temperatures, the action is quadratic and positive, as the temperature increases (i.e. r increases) the quartic term begins to become important and we have a "cowboy hat" shape (see figure 3.4). At this temperature, $\phi_1 = 0$ is no longer a global minimum and bubble nucleation of the deconfined phase sets in. At a critical temperature T_c, the quadratic term in the effective action vanishes, leaving an inverted quartic, and the transition to the deconfined phase is unimpeded. Finally, at temperatures above T_c, the quadratic term reappears, but now with a negative sign. The negative value of the quartic term indicates that the phase transition is indeed first order. The critical temperature is found to be

$$T_c = \frac{\mu}{12\ln(3)}\left[1 + \lambda\frac{2^6\cdot 5}{3^4} - \lambda^2\left(\frac{23\cdot 19927}{2^2\cdot 3^7} + \frac{1765769}{2^4\cdot 3^8}\ln(3)\right) + \dots\right]. \tag{3.95}$$

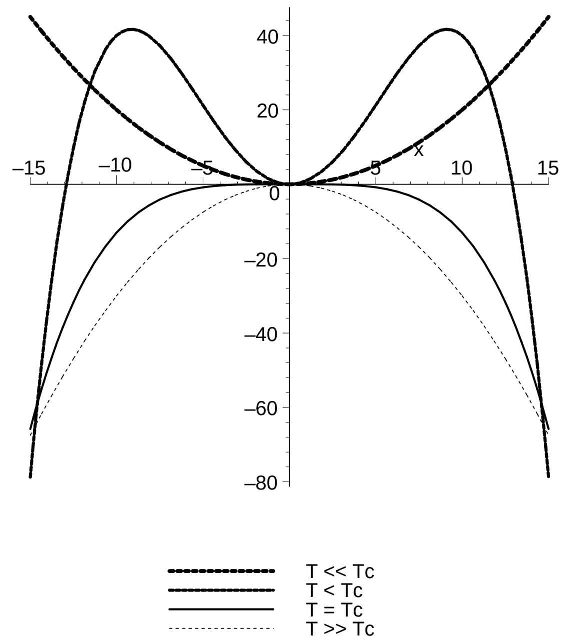


Figure 3.4: The effective action (3.94) for the first Polyakov loop ϕ_1 vs. ϕ_1. At low temperatures, $\phi_1 = 0$, but as the temperature is increased the negative quartic term grows, making this only a local minimum, and leading to bubble nucleation of the deconfined state. Above the critical temperature T_c the quadratic term becomes negative and ϕ_1 condenses to $\phi_1 \neq 0$ indicating total deconfinement.

The zeroth order term in the critical temperature is the one found in [121]. The term of first order in λ agrees with the result quoted in [126].

When r is just less than the (1-loop) critical value $r_{c,0} = 1/3$, the second zero of the effective action is found at

$$|\phi_1|^2 = -\frac{1}{\lambda^2}\frac{\Delta_1(r)}{P_2(r) - P_1^2(r)/\Delta_2(r)}.$$

Higher order terms in the effective action are individually small at this value of $|\phi_1|^2$ when $-\frac{\Delta_1(r)}{P_2(r)-P_1^2(r)/\Delta_2(r)} << 1$. We are further constrained by the fact that $|\phi_1| \leq 1$. This requires that $-\frac{\Delta_1(r)}{P_2(r)-P_1^2(r)/\Delta_2(r)} < \lambda^2 << 1$. The number $-\frac{\Delta_1(r)}{P_2(r)-P_1^2(r)/\Delta_2(r)}$ is less than 0.10 in the range $0.2555 < r \leq 1/3$ and is less than 0.001 in the range $0.3174 < r \leq 1/3$. If r is sufficiently close to $r_{c,0}$, we can reliably say that the absolute minimum of the potential is not at $\phi_1 = 0$ but is elsewhere. This sets an upper bound on the transition temperature

$$T_{\text{trans.}} < T_{c,0} = \frac{\mu}{12\ln(3)}.$$

The tunnelling barrier for bubble nucleation during the first order phase transition is of order $1/\lambda^2$.

3.3.5 Conclusions

We have found that the phase transition in the weakly coupled plane wave matrix model is indeed of first order. As the temperature is raised from zero, the curvature contained in the quadratic term in the effective action still vanishes at some critical temperature. However, before that point is reached, when there is still an energy barrier between the two phases, the deconfined phase becomes the lower energy state. This is the generic behaviour at a first order phase transition. In fact, this behaviour is seen in other adjoint matrix models [130]-[132]. It is also the behaviour that is seen in the collapse of Anti de Sitter space to a black hole, which is thought to be the analog of this phase transition in supergravity of a similar deconfinement in $\mathcal{N}=4$ supersymmetric Yang-Mills theory [112]. It is difficult to speculate what the dual 11-dimensional gravity process may be here. We are working at weak coupling, so that the radius of the five-brane is small compared to the string scale. In other words, the limit of small λ corresponds to a highly curved or "stringy" five-brane geometry [121]. It also corresponds to small R, meaning that we are far from the decompactified M-theory limit. One may speculate that the first order transition persists at strong coupling; if so one would not be surprised to find that it corresponds to the nucleation of black-holes in a classical 11-d SUGRA M5-brane background.

Our analysis does not allow us to compute the first order phase transition temperature accurately, only to deduce that it is of first order. It does, however, allow us to compute corrections to the Hagedorn temperature. This is the temperature at which, if the confining phase is superheated beyond where it is a global minimum of the free energy, it eventually becomes perturbatively unstable. It is just the place where the corrected curvature of the effective action vanishes, i.e. at T_c.

Chapter 4

Exact 1/4 BPS Wilson loop: chiral primary correlator

Show me slowly what I only
know the limits of
— Leonard Cohen *Dance me to the end of love*

One of the most important class of operators in a gauge theory are the Wilson loops. These are non-local, gauge invariant operators which may be thought of as being responsible for the parallel transport of a particle excitation of a field $\psi(x)$, in the fundamental representation, about a closed path C

$$\psi(x+C) = \mathcal{W}\,\psi(x). \tag{4.1}$$

The Wilson loop W is given by, for example in a standard non-abelian gauge theory

$$W = \mathrm{Tr}\,\mathcal{W} = \mathrm{Tr}\,P\exp\left[-i\oint_C dx^\mu A_\mu\right] \tag{4.2}$$

where the trace is over the fundamental representation of the gauge group, and P is the path-ordering symbol, which indicates that in the Taylor expansion of the exponential, higher values of the parameter along the path stand to the left. In section 3.3, we saw that the Wilson loop about the compact thermal circle served as an order parameter for deconfinement. Indeed, the Wilson loop encodes information about the gauge theory generally. If we take C to be a rectangular path in the (t,x) plane, of dimensions $T\times R$, so that the two sides of the rectangle are separated in x by R, then in the limit that $T\gg R$, it is found that

$$\langle W\rangle = A(R)\,e^{-T\,V(R)} \tag{4.3}$$

where $A(R)$ is a numerical pre-factor, and $V(R)$ is the static potential between a fundamental representation particle ψ and its anti-particle, i.e. in QCD, the quark-anti-quark potential. Indeed, the temporal sides of the rectangle may be thought of as the worldlines of the two particles. The fact that they remain straight, despite the interaction between them, indicates that they are effectively treated as infinitely massive.

The string-dual of the Wilson loop in $\mathcal{N}=4$ SYM was one of the first correspondences worked out after the discovery of AdS/CFT [133]. Since then there has been much work on the subject. One of the most significant developments was the work of Erickson, Semenoff, and Zarembo [134]. They discovered that for a circular Wilson loop, an infinite class of diagrams contributing to $\langle W\rangle$ could be summed analytically. The remaining diagrams show a strong promise of cancelling amongst themselves, so that a function is obtained which interpolates smoothly between weak and strong coupling. This is a powerful tool for probing the AdS/CFT correspondence; typically (outside of the BMN double-scaling limit) direct comparison of string results (strong coupling) and CFT results (typically weak coupling) is impossible. This chapter will introduce the Wilson loop in the AdS/CFT correspondence and present original work of the author of this thesis [135] concerning the correlator of a certain 1/4 BPS Wilson loop with chiral primary operators.

4.1 Introduction

In order to construct a Wilson loop in $\mathcal{N} = 4$ SYM, we must find a way to naturally introduce very massive, fundamental particles into the theory. Recall from figure 1.10 that $\mathcal{N} = 4$ SYM is the low-energy worldvolume theory of a stack of N D3-branes. Now consider a stack of $N + 1$ D3-branes, where the extra brane is moved far away from the remaining N, see figure 4.1. This of course corresponds to giving a VEV to the scalar which encodes the transverse

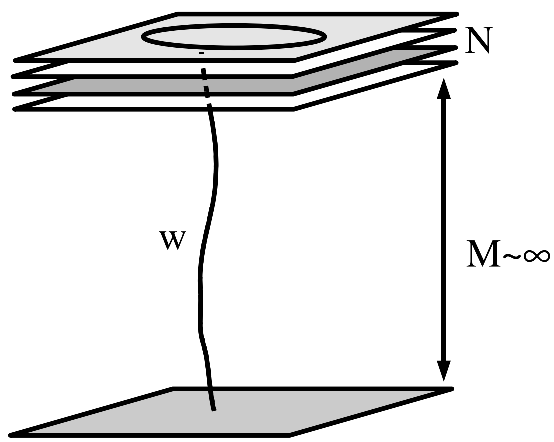


Figure 4.1: Heavy, fundamental particles w may be introduced into $\mathcal{N} = 4$ SYM via a Higgsed $SU(N + 1)$ theory built from a stack of $N + 1$ coincident D3-branes. One brane is placed far from the remaining N, resulting in the introduction of stretched string modes, corresponding to the w field, into the $SU(N)$ theory. The holonomy of such fields about a closed path gives the Wilson loop.

position of the separated brane. In fact [136], the six scalars of the $SU(N + 1)$ theory $\hat{\Phi}^I$ may be expressed as the following $(N + 1) \times (N + 1)$ matrix

$$\hat{\Phi}^I = \begin{pmatrix} \Phi^I & w^I \\ w^{I\dagger} & M\theta^I \end{pmatrix} \tag{4.4}$$

where θ is a constant normalized to $\theta^I\theta^I = 1$, M represents the separation distance of the extra brane, while Φ^I are the scalars of the $SU(N)$ theory. The length-N column field w^I is evidently in the fundamental representation of $SU(N)$. The VEV given to the lower right-hand corner element of $\hat{\Phi}^I$ acts as a Higgs mechanism, imbuing w^I with mass M. From the geometric point of view afforded by the brane construction, the w field is simply those strings stretching from the stack of N D3-branes to the separated brane; their tension is proportional to M. Considering the propagation of a w field about a closed loop (from x^μ to y^μ and then back again) in the $SU(N)$ theory gives the following path-integral representation [136][1]

$$\int dy \langle w(x)\, w^\dagger(x)\, w(y)\, w^\dagger(y)\rangle \sim \int \mathcal{D}x_\mu \int \mathcal{D}A_\mu \mathcal{D}\Phi^I e^{-S_{SU(N)}-ML(x_\mu)} W(x_\mu) \tag{4.5}$$

where on the RHS, the $x_\mu(\tau)$ are closed paths of length $L(x_\mu)$, $S_{SU(N)}$ is the action of $\mathcal{N}=4$ SYM with gauge group $SU(N)$, and the Wilson loop $W(x_\mu)$ is given by

$$W(x_\mu) = \frac{1}{N}\,\mathrm{Tr}\,P\exp\left[\oint_C d\tau\Big(i\dot{x}_\mu(\tau)\,A_\mu(x) + |\dot{x}(\tau)|\theta^I\,\Phi^I(x)\Big)\right]. \tag{4.6}$$

Thus, the Wilson loop measures the holonomy of the w field about a closed path. The scalars of the theory are coupled via $|\dot{x}(\tau)|\theta^I$, where, since $\theta^I\theta^I = 1$, we may interpret θ^I as a point on the five-sphere S^5. More generally, researchers have considered paths on the five-sphere, given by $\theta^I(\tau)$.

The string dual of the $\mathcal{N}=4$ SYM Wilson loop (4.6) was suggested in an early paper by Maldacena [133]. The picture is very intuitive and is summarized in figure 4.2. The Wilson

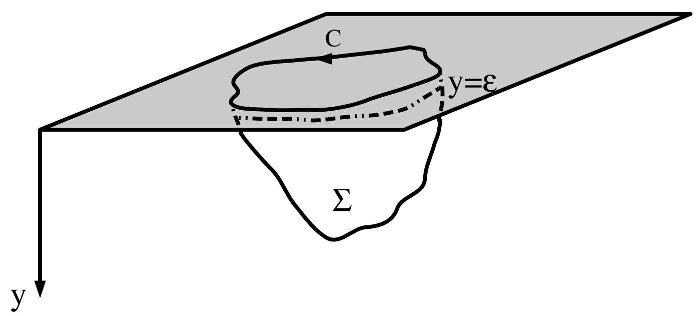


Figure 4.2: The AdS/CFT dual of the Wilson loop (4.6) is given by a macroscopic string whose worldsheet is coincident with the Wilson loop at the boundary of AdS_5.

loop is dual to the semi-classical partition function of a macroscopic string in $AdS_5 \times S^5$, whose worldsheet falls on the path of the Wilson loop at the boundary of AdS_5. More precisely, at strong coupling λ, $\langle W(x_\mu)\rangle$ is given by

$$Z = \int \mathcal{D}X^\mu\,\mathcal{D}Y^I\,\mathcal{D}h_{ab}\mathcal{D}\vartheta^\alpha \exp\Big(-\frac{\sqrt{\lambda}}{2\pi}\int_\Sigma d^2\sigma\sqrt{h}\,h_{ab}\frac{\partial_a X^\mu\partial_b X^\mu + \partial_a Y^I\partial_b Y^I}{Y^2} + \text{fermions}\Big) \tag{4.7}$$

where

[1]As is typical in AdS/CFT, we work in Euclidean signature.

$$X^\mu|_{\partial\Sigma} = x_\mu(\tau), \qquad Y^I|_{\partial\Sigma} = \theta^I(\tau) Y|_{\partial\Sigma}, \qquad Y|_{\partial\Sigma} = 0. \tag{4.8}$$

The saddle-point is obtained when the string worldsheet Σ describes a surface of minimal area $\mathcal{A}$

$$\mathcal{A} = \int_\Sigma d^2\sigma \frac{1}{Y^2}\sqrt{\det\left(\partial_a X^\mu \partial_b X^\mu + \partial_a Y^I \partial_b Y^I\right)} = \mathcal{A}_{\text{reg.}} + \frac{L(C)}{\epsilon} \tag{4.9}$$

where we have noted that such an area is infinite, due to the diverging nature of the area element in AdS_5 as the boundary is approached. In fact, this area may be regulated by placing a cut-off at $Y = \epsilon$. The result is a finite regulated area $\mathcal{A}_{\text{reg.}}$, and an infinite piece proportional to the length of the Wilson loop $L(C)$. This is perfectly consistent with the gauge theory result (4.5), and allows us to associate

$$M \leftrightarrow \frac{1}{\epsilon}. \tag{4.10}$$

We then have that the expectation value of the Wilson loop is given by the exponential of the regulated area of the minimal surface

$$\langle W\rangle = e^{-\frac{\sqrt{\lambda}}{2\pi}\mathcal{A}_{\text{reg.}}}. \tag{4.11}$$

4.1.1 Supersymmetric Wilson loops

Consider as an example the straight line Wilson loop[2] $x_\mu(\tau) = (\tau, 0, 0, 0)$, with θ^I constant. If we calculate the expectation value in perturbation theory, we find

$$\begin{aligned}\langle W\rangle &= 1 + \frac{1}{N}\left\langle \mathrm{Tr}\int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau_2 \left(iA_0 + \theta\cdot\Phi\right)(\tau_1)\left(iA_0 + \theta\cdot\Phi\right)(\tau_2) + \dots\right\rangle \\ &= 1 + \frac{-1 + \theta\cdot\theta}{4\pi^2\left(x(\tau_1) - x(\tau_2)\right)^2} + \dots = 1 + 0 + \dots\end{aligned} \tag{4.12}$$

so the (combined scalar and gauge field) loop-to-loop propagator vanishes for the straight line. In fact this is a special case of a class of supersymmetric Wilson loops, due to Zarembo [137], which have

$$\theta^I(\tau) = \frac{\dot{x}_\mu(\tau)}{|\dot{x}(\tau)|} M^I_\mu, \qquad \text{where} \qquad M^I_\mu M^I_\nu = \delta_{\mu\nu} \tag{4.13}$$

For these loops, the loop-to-loop propagator always vanishes

$$\begin{aligned}\left\langle \left(i\dot{x}_\mu A_\mu + |\dot{x}|\theta\cdot\Phi\right)(\tau_1)\left(i\dot{x}_\mu A_\mu + |\dot{x}|\theta\cdot\Phi\right)(\tau_2)\right\rangle &= \frac{-\dot{x}(\tau_1)\cdot\dot{x}(\tau_2) + M^I_\mu M^I_\nu \dot{x}_\mu(\tau_1)\dot{x}_\nu(\tau_2)}{4\pi^2\left(x(\tau_1) - x(\tau_2)\right)^2} \\ &= 0.\end{aligned}$$

[2]The line is infinite and may be thought of as closing at infinity.

As the name implies, this is a result of supersymmetry. As $d = 4$, $\mathcal{N} = 4$ super-Yang-Mills theory is just a dimensional reduction of $d = 10$, $\mathcal{N} = 1$ super-Yang-Mills theory, we may use the ten dimensional supersymmetry transformations

$$\delta A^\mu = \frac{i}{2}\bar{\epsilon}\gamma^\mu\psi, \quad \delta\Phi^I = \frac{i}{2}\bar{\epsilon}\Gamma^I\psi, \quad \delta\psi = -\frac{1}{4}\Gamma^{MN}F_{MN}\epsilon, \quad \Gamma^{MN} = \frac{1}{2}[\Gamma^M, \Gamma^N] \tag{4.14}$$

where $M = (\mu, I)$ so that $I = 4, ..., 9$, and $\mu = 0, ..., 3$. The 10-d gamma matrices are $\Gamma^M = (\gamma^\mu, \Gamma^I)$ and ψ is a 10-d Majorana-Weyl fermion. The generalized field strength F^{MN} is understood as being built from the 10-d gauge field $A^M = (A^\mu, \Phi^I)$. Further, the structure of ϵ, a 16-component spinor, is as follows

$$\epsilon = \epsilon_0 + x_\mu\gamma^\mu\epsilon_1 \tag{4.15}$$

where ϵ_0 corresponds to the Poincaré, and ϵ_1 to the superconformal supersymmetries. Taking a supersymmetry variation of the Wilson loop gives

$$\delta_\epsilon W = \frac{1}{N}\,\mathrm{Tr}\,P\int d\tau\bar{\psi}\left(i\dot{x}_\mu\gamma^\mu + |\dot{x}|\theta\cdot\Gamma\right)\epsilon\exp\left(\int d\tau'\left(i\dot{x}_\mu A_\mu + |\dot{x}|\theta\cdot\Phi\right)\right). \tag{4.16}$$

Thus, if $(i\dot{x}_\mu\gamma^\mu + |\dot{x}|\theta\cdot\Gamma)\epsilon = 0$ for some constant ϵ, the Wilson loop will enjoy some amount of supersymmetry. In fact this operator is nilpotent, potentially indicating a halving of the supersymmetry

$$(i\dot{x}_\mu\gamma^\mu + |\dot{x}|\theta\cdot\Gamma)^2 = -\dot{x}^2 + \dot{x}^2\,\theta\cdot\theta = 0 \tag{4.17}$$

but, in general solutions will require $\epsilon(\tau)$ which is *local* SUSY - not a symmetry of $\mathcal{N} = 4$ SYM. In the case of the supersymmetric loop, the path dependence factorizes

$$\dot{x}_\mu(\tau)\left(i\gamma^\mu + M^I_\mu\Gamma^I\right)\left(\epsilon_0 + x_\nu(\tau)\gamma^\nu\epsilon_1\right) = 0 \tag{4.18}$$

which gives one halving for each non zero component of $\dot{x}_\mu(\tau)$, and which acts independently on the Poincaré and superconformal supersymmetries. Thus we have that

$$\text{d-dimensional loop} \to \left(\frac{1}{2}\right)^d \quad \text{BPS}. \tag{4.19}$$

These supersymmetric loops, as suggested by the vanishing of the loop-to-loop propagator, have a protected expectation value of exactly unity, independent of the contour $x_\mu(\tau)$

$$\langle W_{\mathrm{SUSY}}\rangle = 1 \qquad \text{(independent of contour)}. \tag{4.20}$$

This has been proven using superspace techniques [138] up to 1/8 BPS loops, while the remaining case of 1/16 BPS was proven in [139]. On the string side of the AdS/CFT duality, this protection has also been proven [140] for the most general case. As a simple example, we may consider the straight line. We expect $\mathcal{A}_{\mathrm{reg.}} = 0$, since $\langle W\rangle = \exp(-\sqrt{\lambda}\mathcal{A}_{\mathrm{reg.}}/(2\pi))$. The string worldsheet sits at a point on the S^5, so we just need the AdS_5 piece

$$\mathcal{A} = \int d\tau d\sigma \frac{1}{Y^2}\sqrt{\left(X'^2 + Y'^2\right)\left(\dot{X}^2 + \dot{Y}^2\right) - \left(X' \cdot \dot{X} + Y'\dot{Y}\right)}. \tag{4.21}$$

If we set $X^{1,2,3} = 0$, we have two embedding functions to worry about, these are $X^0(\sigma, \tau)$ and $Y(\sigma, \tau)$. But we also have this much gauge invariance. Actually the choice

$$X^0(\sigma, \tau) = \sigma \qquad Y(\sigma, \tau) = \tau \tag{4.22}$$

solves the equations of motion trivially and obeys the B.C.'s

$$X^\mu(\sigma, 0) = x_\mu = (\sigma, 0, 0, 0) \qquad Y(\sigma, 0) = 0 \tag{4.23}$$

so then we have

$$\mathcal{A} = \int d\sigma \int_\epsilon^\infty \frac{1}{\tau^2} = \frac{\int d\sigma}{\epsilon} = \frac{L(C)}{\epsilon} \to \mathcal{A}_{\text{reg.}} = 0. \tag{4.24}$$

4.1.2 The 1/2 BPS circle: The straight line's conformal half-brother

One of the most significant Wilson loops to have been considered is the 1/2 BPS circle. This is not a supersymmetric Wilson loop, indeed it cannot be, since a circle has dimension 2, and would therefore be 1/4 BPS by (4.19). The 1/2 BPS circle is given by

$$x_\mu(\tau) = R\,(\cos\tau, \sin\tau, 0, 0), \qquad \theta^I = \text{const.}. \tag{4.25}$$

This Wilson loop has an intimate connection with the supersymmetric straight-line, considered in the last subsection. Indeed the conformal inversion $x_\mu \to x_\mu/x^2$ maps the straight-line to the circle, as shown in figure 4.3. We know that the gauge theory is a conformal field

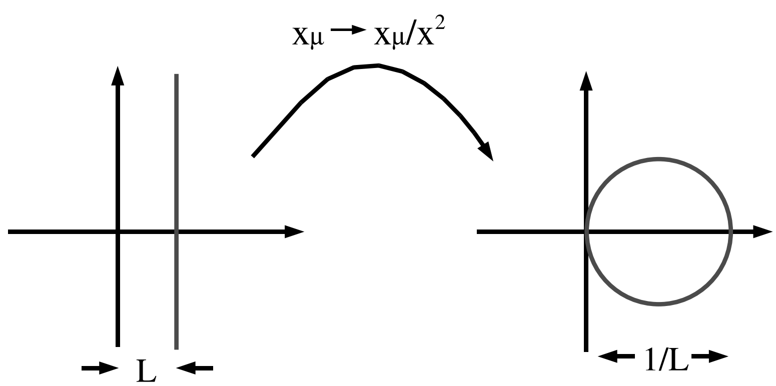


Figure 4.3: The infinite straight line is related to the circle via a conformal inversion. This turns out to be a singular conformal transformation due to the infinite nature of the straight line.

theory, and therefore one might assume that a conformal transformation such as $x_\mu \to x_\mu/x^2$

would not be detectable. However we will see that $\langle W \rangle \neq 1$ for the 1/2 BPS circle. This may be traced to the supersymmetry, which has evidently changed compared to the straight-line

$$(i\dot{x}_\mu \gamma^\mu + \theta \cdot \Gamma)\,(\epsilon_0 + x_\mu \gamma^\mu \epsilon_1) = 0 \qquad \rightarrow \qquad i\gamma^1\gamma^0\epsilon_1 = -\theta \cdot \Gamma \epsilon_0. \tag{4.26}$$

We therefore see that the circle is indeed 1/2 BPS, but as a result of the superconformal and Poincaré supersymmetries being related. In the case of the straight-line, we had two independent conditions on each of ϵ_0 and ϵ_1. We will see below the resolution of this apparent paradox. However, before we do this we will introduce some very important work on the gauge theory calculation of $\langle W \rangle$ for the 1/2 BPS circle.

Erickson, Semenoff, and Zarembo

In the seminal work [134], Erickson, Semenoff, and Zarembo succeeded in summing an infinite class of Feynman diagrams contributing to $\langle W \rangle$, for the 1/2 BPS circle. They noted that the loop-to-loop propagator on the circle is a constant

$$\Big\langle \big(i\dot{x}_\mu A_\mu + \theta \cdot \Phi\big)\big(i\dot{x}_\mu A_\mu + \theta \cdot \Phi\big) \Big\rangle = \frac{1 - \cos\tau_1 \cos\tau_2 - \sin\tau_1 \sin\tau_2}{4\pi^2\left[(\cos\tau_1 - \cos\tau_2)^2 + (\sin\tau_1 - \sin\tau_2)^2\right]} = \frac{1}{8\pi^2} \tag{4.27}$$

So that summing planar ladder diagrams becomes a counting exercise. In figure 4.4, the circular Wilson loop is opened to a horizontal line which is periodically identified. The

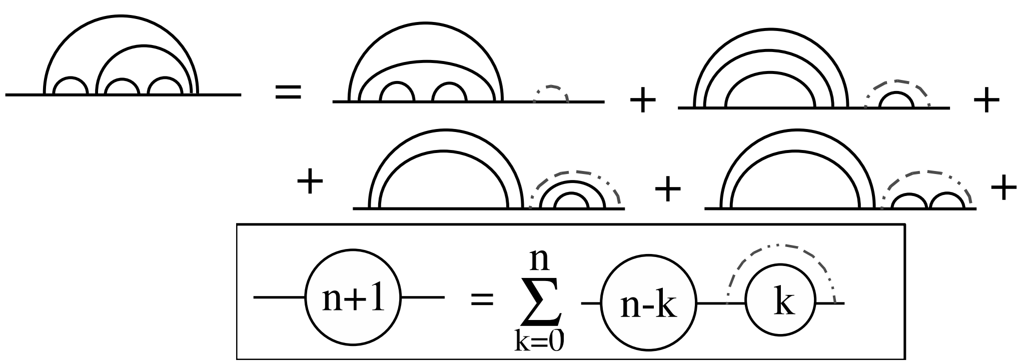


Figure 4.4: Summing the planar ladder diagrams contributing to $\langle W \rangle$ for the 1/2 BPS circle is reduced to a counting exercise owing to the constancy of the loop-to-loop propagator (4.27).

arches represent the loop-to-loop propagators. For example, as shown, all five-propagator (planar, ladder) diagrams may be generated by taking all four-propagator diagrams with a single separated arch (dashed grey line), plus all three propagator diagrams in which all one propagator diagrams are inserted under the separated arch, and so on. This gives a recursion relation for the number of diagrams with n propagators which may then be solved

$$N_{n+1} = \sum_{k=0}^{n} N_{n-k} N_k \qquad \rightarrow \qquad N_n = \frac{(2n)!}{(n+1)!n!} \tag{4.28}$$

Taking care of factors from the path-ordered integration, one finds

$$\langle W \rangle_{\text{ladders}} = \sum_{n=0}^{\infty} \frac{(\lambda/4)^n}{(n+1)!n!} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}). \tag{4.29}$$

The diagrams neglected in this treatment are those with internal vertices. These were shown to cancel amongst themselves, up to two loop order in [134] and [142], and to three loop order in [143]. The minimal area surface of the string dual was found in [141], with the result

$$\langle W \rangle_{\text{string}} = e^{\sqrt{\lambda}}. \tag{4.30}$$

Taking the large-λ limit of (4.29), one finds

$$\langle W \rangle_{\text{gauge}} \simeq \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}} \tag{4.31}$$

and so the same exponential behaviour as the string result. In fact, the presence of the prefactors may also be explained from the string theory perspective. This leads one to suspect that (4.29) is in fact exact, and represents a continuous bridge connecting weak and strong coupling.

Drukker and Gross

In their paper [134], Erickson, Semenoff, and Zarembo also noted that their results could be obtained from a Hermitian matrix model

$$\langle W_{\text{circle}} \rangle = \frac{1}{Z} \int DM \frac{1}{N} \operatorname{Tr} \exp M \exp\left(-\frac{2N}{\lambda} \operatorname{Tr} M^2\right) \tag{4.32}$$

In [144], Drukker and Gross went further with the matrix model and solved also for arbitrary N

$$\langle W_{\text{circle}} \rangle = \frac{1}{N} L^1_{N-1}\left(-\sqrt{\lambda}/4N\right) e^{-\lambda/8N} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + \dots \tag{4.33}$$

where L^m_n is the Laguerre polynomial $L^m_n(x) = 1/n! \exp[x] x^{-m} (d/dx)^n (\exp[-x] x^{n+m})$. They also understood that the inversion $x_\mu \to x_\mu / x^2$ is a singular one, which gives a sort of *conformal anomaly*. The dynamics are captured by a 0-dimensional theory at the point mapped from infinity (see figure 4.5), and this is why the matrix model works. In fact, the result is general

$$\langle W_{\text{closed}} \rangle = F(\lambda, N) \langle W_{\text{open}} \rangle \tag{4.34}$$

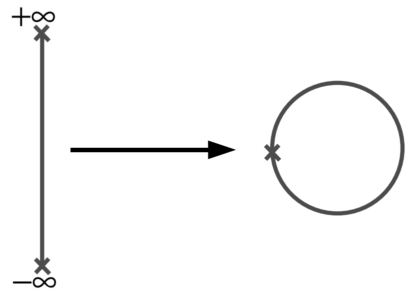


Figure 4.5: Under the conformal inversion $x_\mu \to x_\mu/x^2$, the "end points" of the straight line are mapped to a point on the circle. The dynamics of the straight line Wilson loop trivially vanish by supersymmetry, so that the discrepancy between the two Wilson loops is captured by a 0-dimensional theory living at a point on the circle.

for the relation of any "open" Wilson loop such as the straight-line, to its conformally inverted, closed cousin. The apparent breakdown of conformal invariance is then seen as a consequence of a non-physical infinite Wilson loop, which does not close explicitly.

The discrepancy between (4.30) and (4.31) was also resolved in [144]. They argued that a proper treatment of the semi-classical string partition function should give three powers of $\lambda^{-1/4}$, which should dress the main saddle-point result (4.30). These are associated with the fluctuation determinants of three zero modes associated with the relevant disk amplitude. Drukker and Gross also argued that the disk can be decorated by degenerate handles, which gives an expansion in $1/N$

$$\langle W\rangle_{\text{string}} = \sum_p \frac{C_p}{N^{2p}} \frac{\lambda^{(6p-3)/4}}{p!} e^{\sqrt{\lambda}} \Big(1 + \mathcal{O}(1/\sqrt{\lambda})\Big) \tag{4.35}$$

although the coefficients C_p cannot be easily determined. In fact, a large λ expansion of their matrix model result gives exactly this, with

$$C_p = \sqrt{\frac{2}{\pi}} \frac{1}{96^p}. \tag{4.36}$$

4.1.3 Correlator with a chiral primary operator

When viewed from a large distance, a compact Wilson loop (following the closed path C) should look like an assembly of local operators $\mathcal{O}_{\Delta_i}(x)$ with conformal dimensions Δ_i

$$W[C] = \langle W[C]\rangle \Big(1 + \sum_{\Delta_i>0} \mathcal{O}_{\Delta_i}(0)\ L[C]^{\Delta_i} \xi_{\Delta_i}[C] + \ldots\Big) \tag{4.37}$$

where $L[C]$ is the length of the Wilson loop, and $\xi_{\Delta_i}[C]$ are some coefficients. The leading behaviour of the correlator is given by the operators of smallest conformal dimension - the chiral primaries (c.f. [31]), which we normalize as

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta'}(0)\rangle = \frac{\delta_{\Delta\Delta'}}{(4\pi^2 x^2)^\Delta}. \tag{4.38}$$

We then expect

$$\frac{\langle W[C]\ \mathcal{O}_\Delta(x)\rangle}{\langle W[C]\rangle} = \frac{L[C]^\Delta}{(4\pi^2|x|^2)^\Delta}\xi_\Delta + \dots \tag{4.39}$$

As an example, consider the 1/2 BPS circle with $\theta^I = (1, 0, ..., 0)$

$$\begin{aligned} W &= \langle W\rangle \left(\sum_k (2\pi R)^k \frac{1}{Nk!}\frac{1}{2^k}\operatorname{Tr}\left(Z(0)+\bar{Z}(0)\right)^k + \dots\right) \\ &= \langle W\rangle \left(1 + \sum_{J\geq 2}\mathcal{O}_J(0)(2\pi R)^J \xi_J + \dots\right) \end{aligned} \tag{4.40}$$

where $\mathcal{O}_J(x) = \frac{1}{\sqrt{\lambda^J J}}\operatorname{Tr} Z^J$, $Z = \Phi_1 + i\Phi_2$. We then have, at leading order in λ

$$\frac{\langle \mathcal{O}_J(x)W(0)\rangle}{\langle W(0)\rangle} = \left(\frac{2\pi R}{4\pi^2 x^2}\right)^J \xi_J \qquad \xi_J = \frac{1}{N}\frac{1}{2^J J!}\sqrt{J\lambda^J}. \tag{4.41}$$

In fact, for the 1/2 BPS circle, all planar (loop-to-loop) ladders can also be summed for the calculation of ξ_J. This was accomplished by Semenoff and Zarembo in [145], where leading non-ladder corrections were also found to vanish. The result is

$$\xi_J = \frac{1}{N}\frac{1}{2}\sqrt{\lambda J}\frac{I_J(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \tag{4.42}$$

representing another interpolating bridge between weak and strong coupling, assuming non-ladder diagrams cancel at all orders in perturbation theory.

It is also possible to calculate ξ_J at strong coupling using the string side of the AdS/CFT duality. This was accomplished by Berenstein, Corrado, Fischler, and Maldacena in [141]. The result agrees precisely with the large-λ limit of (4.42). The chiral primaries are dual to supergravitons propagating in $AdS_5 \times S^5$. The large distance correlator (4.39) may be thought of as an exchange of such a mode, between the loop's worldsheet and the boundary of AdS_5, see figure (4.6). For the purpose of calculations, there is an easier method to obtain ξ_J. Berenstein, Corrado, Fischler, and Maldacena pointed out that the leading interaction between a pair of identical but widely separated Wilson loops was mediated by the same supergravitons (see figure 4.7), leading to

$$\frac{\langle W(x)W(0)\rangle}{\langle W(x)\rangle\langle W(0)\rangle} = \sum_J \xi_J^2 \left(\frac{R}{x}\right)^{2J} + \dots \tag{4.43}$$

In practice, this is calculated by coupling the relevant supergravitons to the string worldsheets and using the appropriate bulk-to-bulk propagator. We will give the specific details below, where we demonstrate this calculation for a special class of 1/4 BPS circular loops.

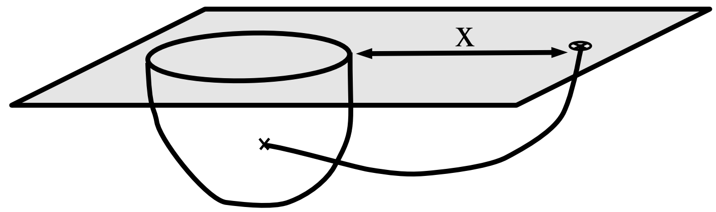


Figure 4.6: The correlator of a Wilson loop with a chiral primary operator (4.39) is dual to the exchange of a supergravity mode between the string worldsheet describing the Wilson loop and the boundary of AdS_5.

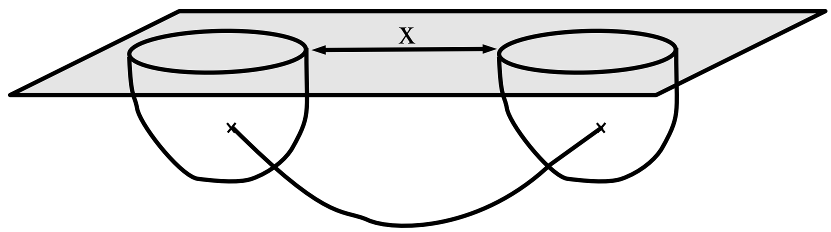


Figure 4.7: A simpler method of calculating ξ_J in (4.41) is to consider the exchange of the same supergravity mode pictured in figure 4.6 between two widely separated Wilson loops.

4.2 Exact 1/4 BPS loop: chiral primary correlator

This section is a presentation of the author's original work published in arXiv:hep-th/0609158 [135].

In a recent paper [146], Drukker proposed and studied the following circular Wilson loop

$$x_\mu(\tau) = R\,(\cos\tau, \sin\tau, 0, 0), \qquad \theta^I(\tau) = (\sin\theta_0\cos\tau, \sin\theta_0\sin\tau, \cos\theta_0, 0, 0, 0). \tag{4.44}$$

When $\theta_0 = \pi/2$, we have the 1/4 BPS SUSY circle of Zarembo, while when $\theta_0 = 0$, the 1/2 BPS circle is recovered. For general θ_0, there is one condition each on ϵ_0 and ϵ_1 (see (4.15)), and one more condition relating them

$$\begin{aligned} \sin\theta_0(\gamma^1\Gamma^2 + \gamma^2\Gamma^1)\epsilon_0 &= 0 \\ \sin\theta_0(\gamma^1\Gamma^2 + \gamma^2\Gamma^1)\epsilon_1 &= 0 \\ \cos\theta_0\epsilon_0 &= R(-i\gamma^1 + \sin\theta_0\Gamma^2)\Gamma^3\gamma^2\epsilon_1 \end{aligned} \tag{4.45}$$

and so the loop is generally 1/4 BPS. The path $\theta^I(\tau)$ describes a circle of latitude θ_0 on an $S^2 \subset S^5$, see figure 4.8. Drukker discovered that, like for the case of the 1/2 BPS circle, the loop-to-loop propagator is a constant $\cos^2\theta_0/8\pi^2$. This is just $\cos^2\theta_0$ times the 1/2 BPS

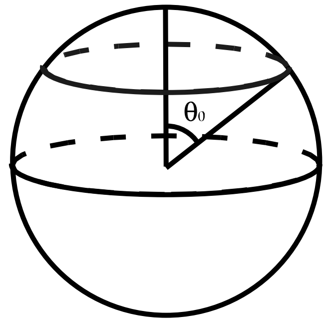


Figure 4.8: The path $\theta^I(\tau)$ (4.44) describes a circle of latitude θ_0 on an $S^2 \subset S^5$.

circle propagator. Therefore, the planar ladder diagrams can be summed in exactly the same way they were for the 1/2 BPS circle. Further, leading internal vertex diagrams cancel in the calculation of $\langle W \rangle$ by the same mechanism as for the 1/2 BPS circle. The only difference is that $\lambda \to \lambda' = \cos^2\theta_0 \lambda$. On the string side, the minimal surface for this 1/4 BPS circle was found by Drukker and it yields $\langle W \rangle = \exp(\sqrt{\lambda'})$. It would thus seem that the results of the 1/2 BPS circle are applicable here, albeit with the rescaled coupling λ'. One therefore expects the matrix model result (4.33) to be applicable here, i.e.

$$\langle W_{1/4} \rangle = \frac{1}{N} L^1_{N-1}\left(-\sqrt{\lambda'}/4N\right) e^{-\lambda'/8N}. \tag{4.46}$$

In the work [135], the author of this thesis and Semenoff expanded the $\lambda \to \lambda'$ correspondence to include correlators with chiral primary operators. That work is described in the balance of this chapter. Certain passages are taken from that publication [135].

4.2.1 Supersymmetry

In the case of the 1/2 BPS circle, the planar ladder diagrams contributing to the correlator with a chiral primary operator are summable and produce (4.42). The remaining diagrams appear to cancel out. The reason for this cancellation is most likely the shared supersymmetry between the chiral primary operator (CPO) and the Wilson loop itself. It is therefore interesting to understand the degree of shared SUSY between the 1/4 BPS circle (4.44) and a generic CPO. We will consider a chiral operator which has an arbitrary $SO(6)$ orientation, beginning with

$$\mathcal{O}(0) = \frac{1}{\sqrt{J\lambda^J}} \mathrm{Tr}\left(u \cdot \Phi(0)\right)^J \tag{4.47}$$

where u is a complex 6-vector, satisfying the constraint that $u^2 = 0$. Being a scalar operator, conformal supersymmetries are automatic. This operator has some Poincaré supersymmetry if there exist some non-zero constant spinors ϵ_0 which solve the equation

$$u \cdot \Gamma \epsilon_0 = 0 \tag{4.48}$$

There are solutions only when $(u \cdot \Gamma)^2 = u^2 = 0$ which, as we have assumed, is the case. Then $u \cdot \Gamma$ is half-rank and there are exactly eight independent non-zero solutions of (4.48).

Now we can ask the question as to whether the eight independent ϵ_0 which solve (4.48) have anything in common with the solutions of ϵ_0 arising from (4.45), i.e. are there spinors which solve both of them? Before we answer this question, let us backtrack to the case of the 1/2 BPS loop geometry. There the top two lines of (4.45) are absent and the spinors must solve the last relation with $\theta_0 = 0$. This simply relates ϵ_1 to ϵ_0, eliminating half of the possible spinors. There are 16 independent solutions of this equation – it is 1/2 BPS. Now, consider a chiral primary operator. Without loss of generality, we can consider the operator $\mathrm{Tr}\,(\Phi_1 + i\Phi_2)^J$. It is supersymmetric if ϵ_0 satisfies the equation

$$\left(\Gamma^1 + i\Gamma^2\right)\epsilon_0 = 0$$

The matrix $\Gamma_1 + i\Gamma_2$ has half-rank, so this requirement eliminates half of the supersymmetries generated by ϵ_0. This leaves eight supersymmetries which commute with both the 1/2 BPS Wilson loop and the 1/2-BPS chiral primary operator. As we mentioned, this high degree of residual joint supersymmetry is thought to be responsible for the fact that, apparently, only ladder diagrams contribute to the asymptotic limit of their correlator.

Returning to the 1/4 BPS loop and chiral primary with general orientation, it is easy to see that there is a simultaneous solution of (4.45) and (4.48) only when one of the following holds:

- $u_1 = u_2 = 0$. We can always do an $SO(6)$ rotation which commutes with the loop operator and sets $(u_4, u_5, u_6) \to (u_4, 0, 0)$. Then, there will be simultaneous solutions of (4.45) and (4.48) only when $u_3 = iu_4$ or when $u_3 = -iu_4$. In both of these cases, there are four solutions, corresponding to 1/8 supersymmetry in common between the chiral primary and the Wilson loop. Up to a constant, the chiral primary operator is $\mathrm{Tr}\,(\Phi_3 + i\Phi_4)^J$ or the complex conjugate $\mathrm{Tr}\,(\Phi_3 - i\Phi_4)^J$.

- $u_3 = u_4 = 0$. There is a solution when $u_1 = \pm iu_2$ and there is also 1/8 supersymmetry. The chiral primary is $\mathrm{Tr}\,(\Phi_1 + i\Phi_2)^J$ or its complex conjugate. In this case, we show in Appendix F.3 that the coefficient ξ_J which is extracted from the long range part of the correlator of this operator and the loop vanishes due to R-symmetry. Thus, for all $J > 0$, the coefficients of $\mathrm{Tr}\,(\Phi_1 + i\Phi_2)^J$ or $\mathrm{Tr}\,(\Phi_1 - i\Phi_2)^J$ in the operator expansion of the 1/4 BPS loop are zero.

- $u_1 = \pm iu_2$. There are two non-zero solutions when $u_3 = iu_4$ or when $u_3 = -iu_4$. This corresponds to 1/16 supersymmetry. There are essentially four operators,

  $$\mathrm{Tr}\left(\chi\left(\Phi_1 + i\Phi_2\right) + \left(\Phi_3 + i\Phi_4\right)\right)^J$$

  plus others with substitutions of $\Phi_1 - i\Phi_2$ or $\Phi_3 - i\Phi_4$. In this case too, because of R-symmetry the contribution with any non-zero power of $(\Phi_1 \pm i\Phi_2)$ will be zero. The coefficient $\xi_J[C_{1/4}]$ for these operators is therefore the same as those for the operator $\mathrm{Tr}\,(\Phi_3 \pm i\Phi_4)^J$.

Thus we see that the interesting quantity where there is some degree of supersymmetry common to both the loop operator and the primary is

$$\xi_J[C_{1/4}] = \lim_{|x|\to\infty} \left(\frac{4\pi^2|x|^2}{2\pi R}\right)^J \frac{1}{\sqrt{J\lambda^J}} \frac{\langle 0|\ W[C_{1/4}]\ \mathrm{Tr}\,(\Phi_3(x) + i\Phi_4(x))^J\ |0\rangle}{\langle 0|\ W[C_{1/4}]\ |0\rangle} \tag{4.49}$$

It is these partially supersymmetric configurations which we expect to have some level of protection from quantum corrections. Indeed, we shall find evidence for this. All other possibilities either vanish, are equivalent to (4.49) or have no supersymmetry at all. The cases with no supersymmetry at all are apparently not protected.

4.2.2 Gauge theory calculation

We will present arguments that the sum of planar ladder diagrams contributing to the correlation function in (4.49) gives a contribution which differs from the one for the 1/2 BPS loop quoted in (4.42) by the simple replacement $\lambda \to \lambda\cos^2\theta_0$, so that the total result is

$$\xi_J[C_{1/4}] = \frac{1}{N}\frac{1}{2}\,\sqrt{\lambda\cos^2\theta_0 J}\,\frac{I_J(\sqrt{\lambda\cos^2\theta_0})}{I_1(\sqrt{\lambda\cos^2\theta_0})}. \tag{4.50}$$

To find this result using Feynman diagrams, we begin with the lowest order diagrams, depicted in figure 4.9. There, each occurrence of the scalar Φ_3 in the composite operator

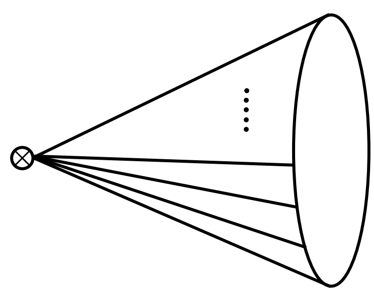


Figure 4.9: The leading planar contribution to $\langle W[C_{1/4}]\ \mathrm{Tr}(\Phi_3 + i\Phi_4)^J\rangle$. There are J lines connecting the chiral primary on the left with the circular Wilson loop on the right.

contracts with a scalar Φ_3 in the Wilson loop. We consider only the planar diagrams. Each scalar Φ_3 from the Wilson loop carries a factor of $\cos\theta_0$, leading to an overall factor of $(\cos\theta_0)^J$. We are taking the convention for Feynman rules where each line in the Feynman diagram results in a factor of λ, totalling λ^J for the diagram in figure 4.9. With this convention, the chiral primary operator has normalization $\lambda^{-J/2}$, as in (4.47). The net result is a factor of $\lambda^{J/2}$ which combines with the $(\cos\theta_0)^J$ to give a coupling constant dependence in the form $(\lambda \cos^2 \theta_0)^{J/2}$. This is identical to what one would have obtained by taking the same diagram for the 1/2 BPS loop and simply replacing λ by $\lambda \cos^2 \theta_0$.

To compute the next orders, we must decorate the diagram in figure 4.9 with propagators. The simplest are ladder diagrams, see figure 4.10, which go between two points on the periphery of the loop. They are described by summing the contribution of the vector and

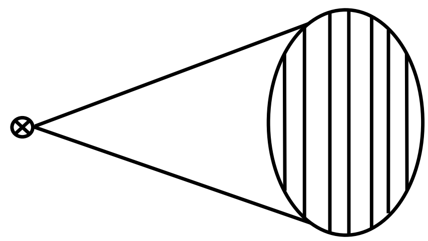


Figure 4.10: A ladder diagram of $\langle W[C_{1/4}]\, \mathrm{Tr}(\Phi_3 + i\Phi_4)^J \rangle$. The "rungs" represent the combined gauge field and scalar propagator. For clarity, J has been set to 2.

the scalar field. Recall that the sum of scalar and vector propagators connecting two points on arcs of the same circle is the constant $\cos^2 \theta_0/(8\pi^2)$. This is what makes ladder diagrams easy to sum. We note that this propagator is accompanied by a factor of λ, so the total λ and θ_0-dependence again comes in the combination $\lambda \cos^2 \theta_0$. Further, the only difference from the analogous quantity for the 1/2 BPS loop is the factor $\cos^2 \theta_0$. Thus we see that the sum of ladders for this 1/4 BPS loop will be identical to that for the 1/2 BPS loop with the replacement $\lambda \to \lambda \cos^2 \theta_0$.

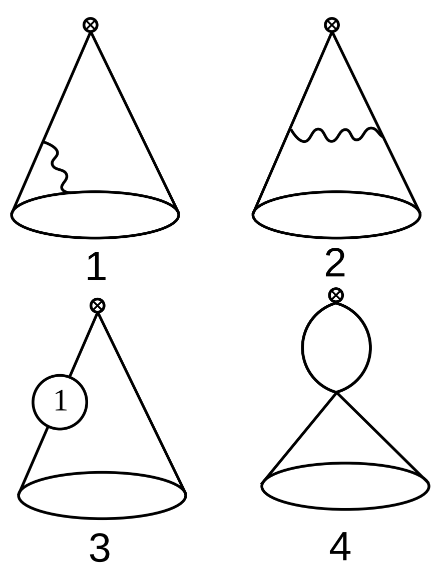


Figure 4.11: The one-loop radiative corrections to $\langle W[C_{1/4}]\, \mathrm{Tr}(\Phi_3 + i\Phi_4)^J \rangle$. Only an adjacent pair of the J scalar lines is shown.

Finally, there are the diagrams that have not yet been included so far. The conjecture is that they vanish. The leading order are depicted in figure 4.11. By a simple generalization of the argument obtained in [145] and explained in more detail in [147], they can be shown to cancel identically. Assuming that this cancellation occurs to higher orders as well, the result for the summation of all planar Feynman diagrams is summarized in the formula (4.50).

4.2.3 String theory calculation

The connected loop-loop correlator (4.43) has an extremal surface whose boundary is the two loops. When the loops have large separation, this surface degenerates to two disc geometry worldsheets whose boundaries are each loop with an infinitesimal tube connecting them, see figure 4.7. In the limit of large separation, this tube is described by the propagator of the lightest gravity modes, which at large λ are 1/2 BPS supergravitons, the string theory duals of the chiral primary operators. The connection between the graviton propagator and the worldsheet is through a vertex operator which must be identified and the connection point with the vertex operator must be integrated over the worldsheet. The resulting amplitude is proportional to the square of the desired operator expansion coefficient, see (4.43).

To begin, the first step is to identify the minimal surface in $AdS_5 \times S^5$ whose boundary is the 1/4 BPS circle $C_{1/4}$. This was done in [146]. We will summarize it here in more convenient coordinates. We take the metric of $AdS_5 \times S^5$

$$\begin{aligned} ds^2 &= \sqrt{\lambda}\left(\frac{dy^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2}{y^2}\right. \\ &+ \left. d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta\left(d\rho^2 + \sin^2\rho\, d\hat{\phi}^2 + \cos^2\rho\, d\tilde{\phi}^2\right)\right). \end{aligned} \tag{4.51}$$

The string worldsheet is then embedded as follows,

$$\begin{aligned} &y = R\tanh\sigma \qquad r_1 = \frac{R}{\cosh\sigma} \qquad \phi_1 = \tau \qquad r_2 = 0 \qquad \phi_2 = \text{const.} \\ &\sin\theta = \frac{1}{\cosh(\sigma_0 \pm \sigma)} \qquad \phi = \tau \qquad \rho = \frac{\pi}{2} \qquad \hat{\phi} = 0 \qquad \tilde{\phi} = \text{const.} \end{aligned} \tag{4.52}$$

where $\sigma \in [0,\infty]$ and $\tau \in [0,2\pi]$ are the worldsheet coordinates. The contour $C_{1/4}$ is the boundary of the worldsheet at $\sigma = 0$, which in turn sits at $y = 0$, the boundary of $AdS_5 \times S^5$. The parameter $\cos\theta_0 = \frac{1}{\cosh\sigma_0}$. The choice of $\pm$ sign in the embedding of θ arises because there are two saddle points in the classical action corresponding to wrapping the north or south pole of the S^5. Of course the sign should be chosen to minimize the classical action, which corresponds to choosing $+$. The other saddle point is unstable, and the string worldsheet will slip-off the unstable pole.

The supergravity modes that we are interested in are fluctuations of the RR 5-form as well as the spacetime metric. They are by now very well known, and details can be found in [148], [141], [32], [150], and [149]. The fluctuations are

$$\begin{aligned} \delta g_{\alpha\beta} &= \left[-\frac{6\,J}{5}\,g_{\alpha\beta} + \frac{4}{J+1}\,D_{(\alpha}D_{\beta)}\right]\,s^J(X)\,Y_J(\Omega), \\ \delta g_{IK} &= 2\,k\,g_{IK}\,s^J(X)\,Y_J(\Omega) \end{aligned} \tag{4.53}$$

where α, β are AdS_5 and I, K are S^5 indices. The symbol X indicates coordinates on AdS^5 and Ω coordinates on the S^5. The $D_{(\alpha}D_{\beta)}$ represents the traceless symmetric double covariant derivative. The $Y_J(\Omega)$ are the spherical harmonics on the five-sphere, while $s^J(X)$ have arbitrary profile and represent a scalar field propagating on AdS_5 space with mass squared $= J(J-4)$, where J labels the representation of $SO(6)$ and must be an integer greater than or equal to 2. (This is the representation of $SO(6)$ which contains the chiral primary operators that we are interested in.)

The supergravity field dual to the operator $\mathrm{Tr}\,(u \cdot \Phi)^J$ is obtained by choosing the combination of spherical harmonics with the same quantum numbers and evaluating them on the worldsheet using (4.52) (see appendix F.2) so that

$$Y_J(\theta, \phi) = \mathcal{N}_J(u) \Big[u_1 \sin\theta\, \cos\phi + u_2 \sin\theta\, \sin\phi + u_3 \cos\theta \Big]^J \tag{4.54}$$

The worldsheets will be connected by the propagator for the scalar supergravity mode $s^J(X)$. The asymptotic form of this propagator for large separation x is

$$P(X, \bar{X}) = \langle s^J(X)\, s^J(\bar{X}) \rangle \simeq \Lambda_J \left(\frac{1}{x} \right)^{2J} y^J\, \bar{y}^J \tag{4.55}$$

where $\Lambda_J = 2^J (J+1)^2/(16\, N^2 J)$. The barred quantities are coordinates on the second Wilson loop worldsheet. Then, in the large λ limit, the Wilson loop correlator is

$$\frac{\langle 0|\; W[C_{1/4}, x]\; W^*[C_{1/4}, 0]\; |0\rangle}{\big|\langle 0|\; W[C_{1/4}]\; |0\rangle\big|^2} = \int_\Sigma \int_{\bar{\Sigma}} \partial_a X^M \partial^a X^N\, \delta g_{MN}\, P(X, \bar{X})\, \delta \bar{g}_{\bar{M}\bar{N}}\, \partial_{\bar{a}} X^{\bar{M}} \partial^{\bar{a}} X^{\bar{N}}, \tag{4.56}$$

where $M, N = 1, ..., 10$ and the δg_{MN} are given in (4.53), except now we have removed the fluctuating parts, $s^J(X)$ and replaced them by the propagator P. The pullback of the fluctuations (4.53) to the worldsheet are found in appendix F.1. Using them we have,

$$\begin{aligned} \frac{\langle 0|\; W[C_{1/4}, x]\; W^*[C_{1/4}, 0]\; |0\rangle}{\big|\langle 0|\; W[C_{1/4}]\; |0\rangle\big|^2} &= \frac{\Lambda_J}{x^{2J}} \frac{\lambda}{16\pi^2} \Bigg[2J \int d\sigma d\tau y'^2 y^{J-2} Y_J(\theta, \phi) - \\ - 2J \int d\sigma d\tau (r_1'^2 + r_1^2) y^{J-2} Y_J(\theta, \phi) &+ 2J \int d\sigma d\tau (\theta'^2 + \sin^2\theta) y^J Y_J(\theta, \phi) \Bigg]^2 \end{aligned} \tag{4.57}$$

Each of the terms inside the square on the right-hand-side of the above expression has a common factor of

$$\int_0^{2\pi} d\tau\, Y_J(\theta, \phi) = \mathcal{N}_J(u) \int_0^{2\pi} d\tau \Big[u_1 \sin\theta \cos\tau + u_2 \sin\theta \sin\tau + u_3 \cos\theta \Big]^J \tag{4.58}$$

From this expression we see that, consistent with our expectations using R-symmetry on the gauge theory side, for the at least 1/16 supersymmetric combination of loop and primary when $u_2 = \pm i u_1$, the dependence on u_1 and u_2 integrates to zero. If these parameters are chosen more arbitrarily, so that there is no supersymmetry at all, the loop depends on them. In that case the contributions proportional to powers of u_1 and u_2 in the final result for the operator expansion coefficients do not follow the rule that they are related to the 1/2 BPS loop ones by the replacement of λ by $\lambda\cos^2\theta_0$. We attribute this to absence of supersymmetry. From here, we will proceed with the supersymmetric case only by putting $u_1 = u_2 = 0$ and $u_3 = 1$.

We will now compute the integrals in (4.57) with this assumption. We note that the embedding (4.52) has some nice properties. For instance $y'^2 + r_1'^2 = r_1^2 = y'$ and also $\sin^2\theta = \theta'^2$. Using these, we can express the integrals in (4.57) as follows

$$\begin{aligned}\frac{2^{-J/2}}{R^J}\int d\sigma y'^2 y^{J-2}\cos^J\theta &= 2^{-J/2}\int_0^\infty d\sigma\frac{(\tanh\sigma)^{J-2}}{\cosh^4\sigma}\tanh^J(\sigma_0\pm\sigma)\\ &= 2^{-J/2}\int_0^1 dz(1-z^2)z^{J-2}\left(\frac{\pm z+\cos\theta_0}{1\pm z\cos\theta_0}\right)^J \end{aligned}\tag{4.59}$$

$$\frac{2^{-J/2}}{R^J}\int d\sigma(r_1'^2+r_1^2)y^{J-2}\cos^J\theta = 2^{-J/2}\int_0^1 dz(1+z^2)z^{J-2}\left(\frac{\pm z+\cos\theta_0}{1\pm z\cos\theta_0}\right)^J \tag{4.60}$$

$$\frac{2^{-J/2}}{R^J}\int d\sigma(\theta'^2+\sin^2\theta)y^J\cos^J\theta = -2^{1-J/2}\int_{\mp\cos\theta_0}^{-1} dz\left(\frac{\pm z+\cos\theta_0}{1\pm z\cos\theta_0}\right)^J z^J \tag{4.61}$$

Putting everything together,

$$\begin{aligned}&\frac{\langle 0|\, W[C_{1/4},x]\, W^*[C_{1/4},0]\, |0\rangle}{\left|\langle 0|\, W[C_{1/4}]\, |0\rangle\right|^2} =\\ &= 16\, J^2\,\frac{\Lambda_J}{2^J}\left(\frac{R}{x}\right)^{2J}\frac{\lambda}{4}\left[\left\{\int_{-1}^{\mp\cos\theta_0} dz - \int_0^1 dz\right\}\left(\frac{\pm z+\cos\theta_0}{1\pm z\cos\theta_0}\right)^J z^J\right]^2\\ &= 16\, J^2\,\frac{\Lambda_J}{2^J}\left(\frac{R}{x}\right)^{2J}\frac{\lambda}{4}\left[\frac{-(\pm)^{J+1}\cos\theta_0}{J+1}\right]^2 = \frac{1}{4N^2}\, J\,\lambda\cos^2\theta_0\left(\frac{R}{x}\right)^{2J}\end{aligned}\tag{4.62}$$

which is just the result for the 1/2 BPS circle [141] with $\lambda\to\lambda\cos^2\theta_0$. Using the prescription (4.43) to obtain from the loop-to-loop correlator the overlap with the chiral primary in question, we find $\xi_J[C_{1/4}] = \sqrt{J\lambda\cos^2\theta_0}/2N$. This is identical to the large λ limit of (4.50). We have thus confirmed that the sum of planar ladder diagrams agrees with the prediction of AdS/CFT in the strong coupling limit. The emergence of this structure on the supergravity side of the duality is non-trivial. The integrations over the AdS_5 and S^5 portions of the string worldsheet conspire in a complicated way in (4.62) to give the $\lambda\to\cos^2\theta_0\,\lambda$ result.

It is instructive to consider this calculation where both saddle points of the classical action are kept in the path integral, as is discussed in [146]. There it was noted that the semi-classical result for the expectation value of the Wilson loop is a sum of two terms; one proportional to $\exp(\sqrt{\lambda'})$ and the other to $\exp(-\sqrt{\lambda'})$, where $\lambda' = \cos^2\theta_0\,\lambda$. This was mirrored in the asymptotic expansion [151] of the modified Bessel function of (4.33)

$$I_1(\sqrt{\lambda'}) =$$
$$\frac{e^{\sqrt{\lambda'}}}{\sqrt{2\pi\sqrt{\lambda'}}} \sum_{k=0}^{\infty} \left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)} \pm i\, \frac{e^{-\sqrt{\lambda'}}}{\sqrt{2\pi\sqrt{\lambda'}}} \sum_{k=0}^{\infty} \left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)} \tag{4.63}$$

where the sign of the i is ambiguous due to the *Stokes' Phenomenon* [152]. The factor of i was associated with the fluctuation determinant of the three tachyonic modes associated with the worldsheet slipping off the unstable pole of the five-sphere.

Due to the sign structure found in (4.62) before squaring, the analogous structure for the connected correlator of the primary with the loop is a sum of a term proportional to $\exp(\sqrt{\lambda'})$ and of another proportional to $(-1)^{J+1}\exp(-\sqrt{\lambda'})$. The sum of these two terms should then be normalized by the expectation value of the Wilson loop. If we employ the asymptotic expansions of the modified Bessel functions in (4.50), we have

$$\frac{I_J(\sqrt{\lambda'})}{I_1(\sqrt{\lambda'})} =$$
$$\frac{e^{\sqrt{\lambda'}} \sum_{k=0}^{\infty} \left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(J+k+1/2)}{k!\,\Gamma(J-k+1/2)} \mp i\,(-1)^J\, e^{-\sqrt{\lambda'}} \sum_{k=0}^{\infty} \left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(J+k+1/2)}{k!\,\Gamma(J-k+1/2)}}{e^{\sqrt{\lambda'}} \sum_{k=0}^{\infty} \left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)} \pm i\, e^{-\sqrt{\lambda'}} \sum_{k=0}^{\infty} \left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)}}. \tag{4.64}$$

This clearly reflects the presence of two saddle points in the functional integrals in both the numerator and denominator.

4.2.4 Summary

The 1/4 BPS circle is quite attractive as it provides a continuous, one parameter family of circular Wilson loops which interpolate between the supersymmetric circle of Zarembo and the celebrated 1/2 BPS circle. Surprisingly, at the level of the Wilson loop expectation value, this entire family of loops seem to be described by the 1/2 BPS circle matrix model, with a rescaled coupling $\lambda' = \cos^2\theta_0 \lambda$, which vanishes for the SUSY circle.

We have presented equal arguments that this correspondence holds for the correlator of the 1/4 BPS circle with a chiral primary operator, as long as that operator shares the minimal 1/16 supersymmetry with the loop. We have found that on the gauge theory side, the planar ladders sum as they do for the 1/2 BPS correlator with a chiral primary. Further, we have shown that the remaining diagrams cancel at leading order. The result is that the $\lambda \to \lambda'$ prescription remains valid. At strong coupling, using string theory, we recover the large-λ' limit of our gauge theory result, as long as the chiral primary in question shares SUSY. We find that when it does not, the $\lambda \to \lambda'$ prescription breaks down. We interpret this as an indication that the correlator is not protected in this case. We therefore expect that higher-order gauge theory calculations will display this lack of protection. It would be very interesting to verify this.

Finally, we note that the double saddle points in the semi-classical action for the string worldsheet describing the 1/4 BPS circle are reflected in our gauge theory results, as was noted in [146] for the expectation value of the loop. It would seem that, as long as a minimum of supersymmetry is maintained, the $\lambda \to \lambda'$ prescription may be extended to include two point functions with chiral primary operators.

Appendix A

Fermion representations

The fermionic normal modes (2.28, 2.29) break the $SO(8)$ symmetry to $SO(4) \times SO(4)$. To make this symmetry manifest it is convenient to label representations of $SO(4)_1 \times SO(4)_2$ through $(SU(2)\times SU(2))_1 \times (SU(2)\times SU(2))_2$ spinor indices. With this decomposition of the R-charge index, the fermionic fields ϑ^a and λ^a, are expressed in terms of creation operators $b^\dagger_{\alpha_1\alpha_2}$ and $b^\dagger_{\dot\alpha_1\dot\alpha_2}$ which transform in the $(1/2, 0, 1/2, 0)$ and $(0, 1/2, 0, 1/2)$ representations of $(SU(2)\times SU(2))_1 \times (SU(2)\times SU(2))_2$, respectively; α_k,$\dot\alpha_k$ being two-component Weyl indices of $SO(4)_k$. The $SO(8)$ vector index I splits into two $SO(4) \times SO(4)$ vector indices (i, i') so that we use vector index $i = 1, \ldots, 4$ and bi-spinor indices $\alpha_1, \dot\alpha_1 = 1, 2$ for the first $SO(4)$ and $(i', \alpha_2, \dot\alpha_2)$ for the second $SO(4)$. Vectors are constructed in terms of bi-spinor indices as $(\alpha_n)_{\alpha_1\dot\alpha_1} = \sigma^i_{\alpha_1\dot\alpha_1}\alpha^i_n/\sqrt{2}$, $(\alpha_n)_{\alpha_2\dot\alpha_2} = \sigma^{i'}_{\alpha_2\dot\alpha_2}\alpha^{i'}_n/\sqrt{2}$ and transform as $(1/2, 1/2, 0, 0)$ and $(0, 0, 1/2, 1/2)$, respectively. Here the σ-matrices consist of the usual Pauli-matrices together with the 2d unit matrix

$$\sigma^i_{\alpha\dot\alpha} = \left(i\tau^1, i\tau^2, i\tau^3, -1\right)_{\alpha\dot\alpha} \tag{A.1}$$

and satisfy the reality properties $\left[\sigma^i_{\alpha\dot\alpha}\right]^\dagger = \sigma^{i\,\dot\alpha\alpha}$, $\left[\sigma^{i\,\dot\alpha}{}_{\alpha}\right]^\dagger = -\sigma^{i\,\alpha}{}_{\dot\alpha}$. These properties are also satisfied by the fermionic oscillators, so that $(\beta_{n\,\alpha_1\,\alpha_2})^\dagger = \beta^{\dagger\,\alpha_1\,\alpha_2}_n$ and $\left(\beta^{\alpha_2}_{n\,\alpha_1}\right)^\dagger = -\beta^{\dagger\,\alpha_1}_{n\,\alpha_2}$; the same relations are obeyed for the dotted-index fermions.

Spinor indices are raised and lowered with the two-dimensional Levi-Civita symbols, $\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $(\epsilon^{\alpha\,\beta})^\dagger = \epsilon_{\beta\,\alpha}$, for example

$$A^\alpha = A_\beta\,\epsilon^{\alpha\,\beta} \qquad A_\alpha = A^\beta\,\epsilon_{\alpha\,\beta} \tag{A.2}$$

and

$$\sigma^i_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,\sigma^{i\,\dot\beta\beta} \equiv \epsilon_{\alpha\beta}\,\sigma^{i\,\beta}{}_{\dot\alpha} \equiv \epsilon_{\dot\alpha\dot\beta}\,\sigma^{i\,\dot\beta}{}_{\alpha}\,. \tag{A.3}$$

The σ-matrices satisfy the relations

$$\sigma^i_{\alpha\dot\alpha}\sigma^{j\,\dot\alpha\beta} + \sigma^j_{\alpha\dot\alpha}\sigma^{i\,\dot\alpha\beta} = 2\delta^{ij}\delta^\beta_\alpha\,, \qquad \sigma^{i\,\dot\alpha\alpha}\sigma^j_{\alpha\dot\beta} + \sigma^{j\,\dot\alpha\alpha}\sigma^i_{\alpha\dot\beta} = 2\delta^{ij}\delta^{\dot\alpha}_{\dot\beta}\,. \tag{A.4}$$

Some other properties satisfied by these matrices are

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = \delta_\alpha^\delta\delta_\beta^\gamma - \delta_\alpha^\gamma\delta_\beta^\delta\,, \tag{A.5}$$

$$\sigma^i_{\alpha\dot\beta}\sigma^{j\,\dot\beta}_{\ \beta} = -\delta^{ij}\epsilon_{\alpha\beta} + \sigma^{ij}_{\alpha\beta}\,, \qquad (\sigma^{ij}_{\alpha\beta} \equiv \sigma^{[i}_{\alpha\dot\alpha}\sigma^{j]\,\dot\alpha}_{\ \beta} = \sigma^{ij}_{\beta\alpha}) \tag{A.6}$$

$$\sigma^i_{\alpha\dot\alpha}\sigma^{j\,\alpha}_{\ \dot\beta} = -\delta^{ij}\epsilon_{\dot\alpha\dot\beta} + \sigma^{ij}_{\dot\alpha\dot\beta}\,, \qquad (\sigma^{ij}_{\dot\alpha\dot\beta} \equiv \sigma^{[i}_{\alpha\dot\alpha}\sigma^{j]\,\alpha}_{\ \dot\beta} = \sigma^{ij}_{\dot\beta\dot\alpha}) \tag{A.7}$$

$$\sigma^k_{\alpha\dot\alpha}\sigma^k_{\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,, \tag{A.8}$$

$$\sigma^{kl}_{\alpha\beta}\sigma^{kl}_{\gamma\delta} = 4(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma})\,, \tag{A.9}$$

$$\sigma^{kl}_{\alpha\beta}\sigma^{kl}_{\dot\gamma\dot\delta} = 0\,, \tag{A.10}$$

$$2\sigma^i_{\alpha\dot\alpha}\sigma^j_{\beta\dot\beta} = \delta^{ij}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \sigma^{k(i}_{\alpha_1\beta_1}\sigma^{j)k}_{\dot\alpha_1\dot\beta_1} - \epsilon_{\alpha\beta}\sigma^{ij}_{\dot\alpha\dot\beta} - \sigma^{ij}_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,. \tag{A.11}$$

$$(\sigma^{ij}_{\dot\alpha\dot\beta})^\dagger = \sigma^{ij\,\dot\alpha\dot\beta} \tag{A.12}$$

$$\sigma^{k\,\dot\beta}_{\gamma}\,\sigma^{i\,j}_{\dot\alpha\,\dot\beta} = \delta^{ik}\sigma^j_{\gamma\,\dot\alpha} - \delta^{jk}\sigma^i_{\gamma\,\dot\alpha} \tag{A.13}$$

In this basis the gamma matrices have the following representation

$$\gamma^i_{a\dot a} = \begin{pmatrix} 0 & \sigma^i_{\alpha_1\dot\beta_1}\delta^{\beta_2}_{\alpha_2} \\ \sigma^{i\,\dot\alpha_1\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} & 0 \end{pmatrix}\,, \qquad \gamma^i_{\dot a a} = \begin{pmatrix} 0 & \sigma^i_{\alpha_1\dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \\ \sigma^{i\,\dot\alpha_1\beta_1}\delta^{\beta_2}_{\alpha_2} & 0 \end{pmatrix}\,, \tag{A.14}$$

$$\gamma^{i'}_{a\dot a} = \begin{pmatrix} -\delta^{\beta_1}_{\alpha_1}\sigma^{i'}_{\alpha_2\dot\beta_2} & 0 \\ 0 & \delta^{\dot\alpha_1}_{\dot\beta_1}\sigma^{i'\,\dot\alpha_2\beta_2} \end{pmatrix}\,, \qquad \gamma^{i'}_{\dot a a} = \begin{pmatrix} -\delta^{\beta_1}_{\alpha_1}\sigma^{i'\,\dot\alpha_2\beta_2} & 0 \\ 0 & \delta^{\dot\alpha_1}_{\dot\beta_1}\sigma^{i'}_{\alpha_2\dot\beta_2} \end{pmatrix}\,. \tag{A.15}$$

and the projector reads

$$\Pi_{ab} = \begin{pmatrix} \left(\sigma^1\sigma^2\sigma^3\sigma^4\right)_{\alpha_1}^{\ \beta_1}\delta^{\beta_2}_{\alpha_2} & 0 \\ 0 & \left(\sigma^1\sigma^2\sigma^3\sigma^4\right)^{\dot\alpha_1}_{\ \dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \end{pmatrix} = \begin{pmatrix} \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} & 0 \\ 0 & -\delta^{\dot\alpha_1}_{\dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \end{pmatrix}\,, \tag{A.16}$$

so that $(1 \pm \Pi)/2$ projects onto $(1/2, 0, 1/2, 0)$ and $(0, 1/2, 0, 1/2)$, respectively.

The supercharge $Q^-_{\alpha_1\dot\beta_2}$ is a $(1/2, 0, 0, 1/2)$ and $Q^-_{\dot\alpha_1\beta_2}$ is a $(0, 1/2, 1/2, 0)$ representation. In this notation it is convenient to define the linear combinations of the free supercharges

$$\sqrt{2}\eta Q \equiv Q^- + i\bar Q^- \quad , \quad \sqrt{2}\bar\eta\widetilde{Q} \equiv Q^- - i\bar Q^- \tag{A.17}$$

where $\eta = e^{i\pi/4}$, and $\bar Q^\pm = e(\alpha)(Q^\pm)^\dagger$. On the space of physical states they satisfy the dynamical constraints

$$\begin{aligned} &\left\{Q_{\alpha_1\dot\alpha_2}, Q_{\beta_1\dot\beta_2}\right\} = \left\{\widetilde{Q}_{\alpha_1\dot\alpha_2}, \widetilde{Q}_{\beta_1\dot\beta_2}\right\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}H \\ &\left\{Q_{\alpha_1\dot\alpha_2}, \widetilde{Q}_{\beta_1\dot\beta_2}\right\} = -\mu\epsilon_{\dot\alpha_2\dot\beta_2}\left(\sigma^{ij}\right)_{\alpha_1\beta_1}J^{ij} + \mu\epsilon_{\alpha_1\beta_1}\left(\sigma^{i'j'}\right)_{\dot\alpha_2\dot\beta_2}J^{i'j'} \end{aligned} \tag{A.18}$$

and similarly for $Q_{\dot\alpha_1\alpha_2}$ and $\widetilde{Q}_{\dot\beta_1\beta_2}$. The free supercharge with raised indices is understood as

$$Q_2^{\alpha_1\dot\alpha_2} \equiv e(\alpha)\,(Q_{2\,\alpha_1\dot\alpha_2})^\dagger\,, \qquad Q_2^{\dot\alpha_1\alpha_2} \equiv e(\alpha)\,(Q_{2\,\dot\alpha_1\alpha_2})^\dagger \tag{A.19}$$

and this gives

$$Q_2^{\alpha_1\dot\alpha_2} Q_{2\,\alpha_1\dot\alpha_2} = +4H_2 = Q_{2\,\alpha_1\dot\alpha_2} Q_2^{\alpha_1\dot\alpha_2} \tag{A.20}$$

for these operators in the single string Hilbert space $\mathcal{H}_1$. For states in the three-string Hilbert space $\mathcal{H}_3$, i.e. $|Q_3\rangle$, the $e(\alpha)$ is already encoded into the construction so that it should be dropped in the adjoint

$$Q_{2\,\alpha_1\dot\alpha_2}|Q_3^{\alpha_1\dot\alpha_2}\rangle = Q_2^{\alpha_1\dot\alpha_2}|Q_{3\alpha_1\dot\alpha_2}\rangle \equiv (Q_{2\,\alpha_1\dot\alpha_2})^\dagger\,|Q_{3\alpha_1\dot\alpha_2}\rangle = +4|H_3\rangle \tag{A.21}$$

and similarly $Q_3^{\alpha_1\dot\alpha_2} \equiv (Q_{3\,\alpha_1\dot\alpha_2})^\dagger$. In the BMN basis, the full expression for the quadratic supercharge $Q_{2\,\alpha_1\,\dot\alpha_2}$ is[1]

$$\begin{aligned} Q_{2\,\alpha_1\,\dot\alpha_2} = {} & \frac{\bar\eta}{\sqrt{|\alpha|}} \sum_{k\neq 0} \Omega_k \Big(\alpha^{\dagger\,\dot\beta_1}_{k\alpha_1}\, \beta_{k\dot\beta_1\,\dot\alpha_2} + i\,e(\alpha)\,\alpha_{k\alpha_1}^{\ \dot\beta_1}\,\beta^\dagger_{k\dot\beta_1\,\dot\alpha_2} \\ & \qquad\qquad + i\,\alpha^{\dagger\,\beta_2}_{k\dot\alpha_2}\,\beta_{k\alpha_1\,\beta_2} + e(\alpha)\,\alpha_{k\dot\alpha_2}^{\ \beta_2}\,\beta^\dagger_{k\alpha_1\,\beta_2}\Big) \\ & + \bar\eta^{\,e(\alpha)}\,\sqrt{2\,\mu}\,\Big(\alpha^{\dagger\,\dot\beta_1}_{0\alpha_1}\,\beta_{0\dot\beta_1\,\dot\alpha_2} + i\,\alpha_{0\alpha_1}^{\ \dot\beta_1}\,\beta^\dagger_{0\dot\beta_1\,\dot\alpha_2} \\ & \qquad\qquad + i\,e(\alpha)\,\alpha^{\dagger\,\beta_2}_{0\dot\alpha_2}\,\beta_{0\alpha_1\,\beta_2} + e(\alpha)\,\alpha_{0\dot\alpha_2}^{\ \beta_2}\,\beta^\dagger_{0\alpha_1\,\beta_2}\Big) \end{aligned} \tag{A.22}$$

where Ω_k is defined in (C.6).

Among states that are created by two oscillators, the state with quantum numbers $(1,1,0,0)$ and $(0,0,1,1)$ which are created by two bosons have no analogues amongst the two oscillator states containing either one or two fermions. Thus, they are not mixed with other members of the supermultiplet. These states in the main text are denoted $\big|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ and $|[\mathbf{1},\mathbf{9}]\rangle^{(i'j')}$ in SO(8) notation.

[1] Note that $\alpha^{\dagger\,\dot\beta_1}_{k\alpha_1} = -\sigma^{i\,\dot\beta_1}_{\ \ \alpha_1}\alpha^{i\,\dagger}_k/\sqrt{2}$, and similarly for the other $SO(4)$ since $[\sigma^{i\,\alpha_1}_{\ \ \dot\beta_1}]^\dagger = -\sigma^{i\,\dot\beta_1}_{\ \ \alpha_1}$.

Appendix B

Neumann matrices and associated quantities

In this section we present the explicit expressions for the quantities appearing in the prefactors and exponential part of $|H_3\rangle$ and $|Q_3\rangle$ (2.68). Following the notation of [86], the Neumann matrices can be written as

$$\widetilde{N}^{st}_{mn} = \begin{cases} \frac{1}{2}\bar{N}^{st}_{|m||n|}\left(1 + U_{m(s)}U_{n(t)}\right) & , m,n \neq 0 \\ \frac{1}{\sqrt{2}}\bar{N}^{st}_{|m|0} \quad m \neq 0 \\ \bar{N}^{st}_{00} \end{cases} \tag{B.1}$$

with[1]

$$\bar{N}^{st}_{mn} = -(1-4\mu\kappa K)^{-1}\frac{\kappa}{\alpha_s\omega_{n(t)} + \alpha_t\omega_{m(s)}}\left[CU^{-1}_{(s)}C^{1/2}_{(s)}\bar{N}^s\right]_m\left[CU^{-1}_{(t)}C^{1/2}_{(t)}\bar{N}^t\right]_n \tag{B.2}$$

$$\bar{N}^{st}_{m0} = \sqrt{-2\mu\kappa(1-\beta_t)}\sqrt{\omega_{m(s)}}\bar{N}^s_m\,, \qquad t \in \{1,2\} \tag{B.3}$$

$$\bar{N}^{st}_{00} = (1-4\mu\kappa K)\left(\delta^{st} + \sqrt{\beta_s\beta_t}\right)\,, \qquad s,t \in \{1,2\} \tag{B.4}$$

$$\bar{N}^{s3}_{00} = -\sqrt{\beta_s}\,, \qquad s \in \{1,2\} \tag{B.5}$$

while

$$\widetilde{Q}^{rs}_{mn} = \begin{cases} \frac{i}{2}e(m)\bar{Q}^{rs}_{|m||n|}, & m,n \neq 0 \\ \frac{i}{\sqrt{2}}e(m)\bar{Q}^{rs}_{|m|0}, & m \neq 0 \\ \bar{Q}^{rs}_{00} \end{cases} \tag{B.6}$$

where [84]

$$\begin{aligned} \bar{Q}^{rs}_{mn} &= e(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}\left[U^{1/2}_{(r)}C^{1/2}\bar{N}^{rs}C^{-1/2}U^{1/2}_{(s)}\right]_{mn}, \qquad m,n>0 \\ \bar{Q}^{sr}_{m0} &= -\alpha_3(1-\beta_r)\sqrt{\alpha_r}\frac{e(\alpha_s)}{\sqrt{|\alpha_s|}}\left[\left(U_{(s)}C_{(s)}C\right)^{1/2}\bar{N}^s\right]_m, \qquad m>0 \\ \bar{Q}^{3r}_{00} &= -\bar{Q}^{r3}_{00} = \frac{1}{2}\sqrt{-\frac{\alpha_r}{\alpha_3}}, \qquad \bar{Q}^{rs}_{00} = 0, \qquad r,s = \{1,2\} \end{aligned} \tag{B.7}$$

[1] To have a manifest symmetry in $1 \leftrightarrow 2$ we additionally redefined the oscillators as $(-1)^{s(n+1)}\alpha_{n(s)} \to \alpha_{n(s)}$ for $n \in \mathbb{Z}$, $s = 1,2,3$ and analogously for the fermionic oscillators.

and we note that $\bar{Q}^{sr}_{0m} = 0$, while

$$C_n = n\,, \qquad C_{n(s)} = \omega_{n(s)} \equiv \sqrt{n^2 + \big(\mu\alpha_s\big)^2}\,, \qquad \kappa \equiv \alpha_1\alpha_2\alpha_3 \tag{B.8}$$

$$U_{n(s)} = \frac{1}{n}(\omega_{n(s)} - \mu\alpha_s)\,, \qquad U^{-1}_{n(s)} = \frac{1}{n}(\omega_{n(s)} + \mu\alpha_s) \tag{B.9}$$

and [80]

$$1 - 4\mu\kappa K \approx -\frac{1}{4\pi r(1-r)\mu\alpha_3} \tag{B.10}$$

$$\alpha_3 \bar{N}^3_n \approx -\frac{\sin(n\pi r)}{\pi r(1-r)} \frac{1}{\omega_{n(3)}\sqrt{-2\mu\alpha_3(\omega_{n(3)} + \mu\alpha_3)}} \tag{B.11}$$

$$\alpha_3 \bar{N}^s_n \equiv \alpha_3 \bar{N}_n(\beta_s) \approx -\frac{\sqrt{\beta_s}}{2\pi r(1-r)} \frac{1}{\omega_{n(s)}\sqrt{-2\mu\alpha_3(\omega_{n(s)} - \mu\alpha_3\beta_s)}} \tag{B.12}$$

up to exponential corrections $\sim \mathcal{O}(e^{-\mu\alpha_3})$ [2]. For the bosonic constituents of the prefactor one has

$$K^I = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} K_{n(s)}\alpha^{I\,\dagger}_{n(s)}\,, \quad \widetilde{K}^I = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} K_{n(s)}\alpha^{I\,\dagger}_{-n(s)} \tag{B.13}$$

where

$$K_{0(s)} = (1-4\mu\kappa K)^{1/2}\sqrt{-\frac{2\mu\kappa}{\alpha'}\big(1-\beta_s\big)}\,, \qquad K_{0(3)} = 0 \tag{B.14}$$

and

$$K_{n(s)} = -\frac{\kappa}{\sqrt{2\alpha'}\alpha_s}(1-4\mu\kappa K)^{-1/2}(\omega_{n(s)} + \mu\alpha_s)\sqrt{\omega_{n(s)}}\bar{N}^s_{|n|}\big(1 - U_{n(s)}\big) \tag{B.15}$$

For the fermionic constituents of the prefactor one has

$$Y^{\alpha_1\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} G_{|n|(s)}\beta^{\dagger\,\alpha_1\alpha_2}_{n(s)}\,, \qquad Z^{\dot\alpha_1\dot\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} G_{|n|(s)}\beta^{\dagger\,\dot\alpha_1\dot\alpha_2}_{n(s)}\,, \tag{B.16}$$

where

$$G_{0(s)} = (1-4\mu\kappa K)^{1/2}\sqrt{1-\beta_s}\,, \qquad G_{0(3)} = 0 \tag{B.17}$$

and

$$G_{n(s)} = \frac{e(\alpha_s)}{\sqrt{2|\alpha_s|}} \frac{\sqrt{-\kappa}}{(1-4\mu\kappa K)^{1/2}} \sqrt{(\omega_{n(s)} + \mu\alpha_s)\omega_{n(s)}}\bar{N}^s_{|n|} \tag{B.18}$$

where in the above expressions we have used $\beta_1 \equiv r$ and $\beta_2 \equiv 1-r$ (with $\beta_t \equiv -\alpha_t/\alpha_3$ and $\alpha_3 < 0$).

[2] To compare with the definition used in [80] note that $\bar{N}^s_{n\,\text{here}} = (-1)^{s(n+1)}U_{n(s)}C^{-1/2}_{n(s)}\bar{N}^s_{n\,\text{there}}$.

Appendix C

Simpler forms and relations

We find a simpler expression for the Neumann matrices and associated quantities

$$\widetilde{N}^{3\,r}_{n\,q} = -\frac{\sin(n\pi r)\sqrt{\beta_r}\left(\Lambda^+_n\Lambda^+_q + \Lambda^-_n\Lambda^-_q\right)}{2\pi\sqrt{\omega_n\omega_q}\,(q-\beta_r n)} \qquad \widetilde{N}^{r\,s}_{q\,p} = \frac{\sqrt{\beta_r\beta_s}\left(\Lambda^+_q\Lambda^+_p + \Lambda^-_q\Lambda^-_p\right)}{4\pi\sqrt{\omega_q\omega_p}\,(\beta_s\omega_q + \beta_r\omega_p)} \tag{C.1}$$

$$\widehat{Q}^{3\,r}_{n\,q} = \frac{i\sin(|n|\pi r)\,(\omega_q + \beta_r\omega_n)}{2\pi\sqrt{\omega_n\omega_q}\,(q-\beta_r n)} \qquad \widehat{Q}^{r\,s}_{q\,p} = \frac{i\,(\beta_s q - \beta_r p)}{4\pi\sqrt{\omega_q\omega_p}\,(\beta_s\omega_q + \beta_r\omega_p)} \tag{C.2}$$

where $\widehat{Q} = \widetilde{Q} - \widetilde{Q}^T$. We also find

$$K_n = +\alpha_3\sin(n\pi r)\sqrt{\frac{r(1-r)}{\pi\alpha'}}\frac{\Lambda^-_n - \Lambda^+_n}{\sqrt{\omega_n}} \tag{C.3}$$

$$K_q = -\alpha_3\sqrt{\frac{r(1-r)}{\pi\alpha'\beta_r}}\frac{\Lambda^+_q - \Lambda^-_q}{2\sqrt{\omega_q}} \tag{C.4}$$

$$G_q = \frac{1}{\sqrt{4\pi\omega_q}} \qquad G_n = -\frac{\sin(|n|\pi r)}{\sqrt{\pi\omega_n}} \tag{C.5}$$

$$\Omega_q = \Lambda^+_q - \Lambda^-_q \qquad \Omega_n = e(n)(\Lambda^-_n - \Lambda^+_n) \tag{C.6}$$

where,

$$\Lambda^+_q = \sqrt{\omega_q - \beta_r\mu\alpha_3} \qquad \Lambda^-_q = e(q)\sqrt{\omega_q + \beta_r\mu\alpha_3} \tag{C.7}$$

$$\Lambda^+_n = \sqrt{\omega_n - \mu\alpha_3} \qquad \Lambda^-_n = e(n)\sqrt{\omega_n + \mu\alpha_3} \tag{C.8}$$

We will also find use for

$$L^{3\,r}_{n\,q} \equiv K_n K_{-q} + K_{-n}K_q \qquad \widetilde{L}^{3\,r}_{n\,q} \equiv K_n K_q + K_{-n}K_{-q}. \tag{C.9}$$

The following relations may also be proven

$$K^{(s)}_p\,K^{(r)}_q + K^{(s)}_{-p}\,K^{(r)}_{-q} = \frac{2\,\alpha_3^2\,r\,(1-r)}{\alpha'}\left(\frac{\omega^{(r)}_q}{\beta_r} + \frac{\omega^{(s)}_p}{\beta_s}\right)\widetilde{N}^{r\,s}_{q\,p} \tag{C.10}$$

$$K^{(3)}_n\,K^{(r)}_q + K^{(3)}_{-n}\,K^{(r)}_{-q} = \frac{2\,\alpha_3^2\,r\,(1-r)}{\alpha'}\left(\frac{\omega^{(r)}_q}{\beta_r} - \omega^{(3)}_n\right)\widetilde{N}^{3\,r}_{n\,q} \tag{C.11}$$

$$\Omega_q^{(r)}\, G_q^{(r)} = \sqrt{\frac{\beta_r\,\alpha'}{r\,(1-r)}}\,\frac{1}{-\alpha_3}\,K_q^{(r)} \tag{C.12}$$

$$\Omega_n^{(3)}\, G_n^{(3)} = \sqrt{\frac{\alpha'}{r\,(1-r)}}\,\frac{1}{-\alpha_3}\,K_n^{(3)} \tag{C.13}$$

$$i\,\frac{\Omega_q^{(r)}}{\sqrt{\beta_r}}\,\widehat{Q}^{r\,s}_{q\,p} + \frac{\Omega_p^{(s)}}{\sqrt{\beta_s}}\,\widetilde{N}^{r\,s}_{q\,p} = \sqrt{\frac{\alpha'}{r\,(1-r)}}\,\frac{1}{-\alpha_3} K_q^{(r)} G_p^{(s)} \tag{C.14}$$

$$i\,\Omega_n^{(3)}\,\widehat{Q}^{3\,r}_{n\,q} + \frac{\Omega_q^{(r)}}{\sqrt{\beta_r}}\,\widetilde{N}^{3\,r}_{n\,q} = \sqrt{\frac{\alpha'}{r\,(1-r)}}\,\frac{1}{-\alpha_3} K_n^{(3)} G_q^{(r)} \tag{C.15}$$

$$-i\,\frac{\Omega_q^{(r)}}{\sqrt{\beta_r}}\,\widehat{Q}^{3\,r}_{n\,q} + \Omega_n^{(3)}\,\widetilde{N}^{3\,r}_{n\,q} = \sqrt{\frac{\alpha'}{r\,(1-r)}}\,\frac{1}{-\alpha_3} K_q^{(r)} G_n^{(3)} \tag{C.16}$$

$$\left(\widetilde{N}^{r\,s}_{q\,p}\right)^2 - \left(\widehat{Q}^{r\,s}_{q\,p}\right)^2 = \left(G_{|q|}^{(r)}\,G_{|p|}^{(s)}\right)^2 \tag{C.17}$$

$$\left(\widetilde{N}^{3\,r}_{n\,q}\right)^2 + \left(\widehat{Q}^{3\,r}_{n\,q}\right)^2 = -\left(G_{|q|}^{(r)}\,G_{|n|}^{(3)}\right)^2 \tag{C.18}$$

Appendix D

Calculational method

D.1 Vertices and definitions

We remind the reader of the construction of $|H_3\rangle$ and $|Q_3\rangle$ in (2.68)

$$|V\rangle = |E_\alpha\rangle|E_\beta\rangle\delta(\sum_{r=1}^{3}\alpha_r) \tag{D.1}$$

where $|E_\alpha\rangle$ and $|E_\beta\rangle$ are exponentials of bosonic and fermionic oscillators respectively

$$|E_\alpha\rangle = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\alpha^{\dagger K}_{m\,(s)}\widetilde{N}^{st}_{mn}\alpha^{\dagger K}_{n\,(t)}\right)|\alpha\rangle_{123} \tag{D.2}$$

and

$$|E_\beta\rangle = \exp\left(\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\big(\beta^{\alpha_1\alpha_2\,\dagger}_{m(r)}\beta^{\dagger}_{n(s)\,\alpha_1\alpha_2} - \beta^{\dot\alpha_1\dot\alpha_2\,\dagger}_{m(r)}\beta^{\dagger}_{n(s)\,\dot\alpha_1\dot\alpha_2}\big)\widetilde{Q}^{rs}_{mn}\right)|\alpha\rangle_{123} \tag{D.3}$$

where $|\alpha\rangle_{123} = |0;\alpha_1\rangle\otimes|0;\alpha_2\rangle\otimes|0;\alpha_3\rangle$. We then have

$$\begin{aligned}
|H_3\rangle &= g_2\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{\alpha'}{8\,\alpha_3^3}\Big[\big(K_i\widetilde{K}_j - \frac{\mu\kappa}{\alpha'}\delta_{ij}\big)v^{ij} - \big(K_{i'}\widetilde{K}_{j'} - \frac{\mu\kappa}{\alpha'}\delta_{i'j'}\big)v^{i'j'} \\
&\quad - K^{\dot\alpha_1\alpha_1}\widetilde{K}^{\dot\alpha_2\alpha_2}s_{\alpha_1\alpha_2}(Y)s^*_{\dot\alpha_1\dot\alpha_2}(Z) - \widetilde{K}^{\dot\alpha_1\alpha_1}K^{\dot\alpha_2\alpha_2}s^*_{\alpha_1\alpha_2}(Y)s_{\dot\alpha_1\dot\alpha_2}(Z)\Big]|V\rangle\,, \\
|Q_{3\,\beta_1\dot\beta_2}\rangle &= g_2\,\eta\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s_{\dot\gamma_1\dot\beta_2}(Z)t_{\beta_1\gamma_1}(Y)\widetilde{K}^{\dot\gamma_1\gamma_1} \\
&\qquad\qquad + is_{\beta_1\gamma_2}(Y)t^*_{\dot\beta_2\dot\gamma_2}(Z)\widetilde{K}^{\dot\gamma_2\gamma_2}\Big)|V\rangle\,, \\
|Q_{3\,\dot\beta_1\beta_2}\rangle &= g_2\,\bar\eta\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s^*_{\gamma_1\beta_2}(Y)t^*_{\dot\beta_1\dot\gamma_1}(Z)\widetilde{K}^{\dot\gamma_1\gamma_1} \\
&\qquad\qquad + is^*_{\dot\beta_1\dot\gamma_2}(Z)t_{\beta_2\gamma_2}(Y)\widetilde{K}^{\dot\gamma_2\gamma_2}\Big)|V\rangle\,.
\end{aligned} \tag{D.4}$$

where

$$K^I = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}}K_{n(s)}\alpha^{I\,\dagger}_{n(s)}\,, \quad \widetilde{K}^I = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}}K_{n(s)}\alpha^{I\,\dagger}_{-n(s)} \tag{D.5}$$

$$Y^{\alpha_1\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} G_{|n|(s)}\beta^{\dagger\,\alpha_1\alpha_2}_{n(s)}\,, \qquad Z^{\dot\alpha_1\dot\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbb{Z}} G_{|n|(s)}\beta^{\dagger\,\dot\alpha_1\dot\alpha_2}_{n(s)}\,, \tag{D.6}$$

and

$$\widetilde{K}^{\dot\gamma_1\gamma_1} \equiv \widetilde{K}^i\sigma^{i\,\dot\gamma_1\gamma_1}\,, \qquad \widetilde{K}^{\dot\gamma_2\gamma_2} \equiv \widetilde{K}^{i'}\sigma^{i'\,\dot\gamma_2\gamma_2}\,, \tag{D.7}$$

where the σ-matrices are defined in appendix A. We also have

$$\begin{aligned} v^{ij} &= \delta^{ij}\Big[1 + \frac{1}{12}\big(Y^4+Z^4\big) + \frac{1}{144}Y^4Z^4\Big] \\ &\quad - \frac{i}{2}\Big[Y^{2ij}\big(1+\frac{1}{12}Z^4\big) - Z^{2ij}\big(1+\frac{1}{12}Y^4\big)\Big] + \frac{1}{4}\big[Y^2Z^2\big]^{ij}\,, \\ v^{i'j'} &= \delta^{i'j'}\Big[1 - \frac{1}{12}\big(Y^4+Z^4\big) + \frac{1}{144}Y^4Z^4\Big] \\ &\quad - \frac{i}{2}\Big[Y^{2i'j'}\big(1-\frac{1}{12}Z^4\big) - Z^{2i'j'}\big(1-\frac{1}{12}Y^4\big)\Big] + \frac{1}{4}\big[Y^2Z^2\big]^{i'j'}\,. \end{aligned}$$

Here we defined

$$Y^{2ij} \equiv \sigma^{ij}_{\alpha_1\beta_1}Y^{2\alpha_1\beta_1}\,, \quad Z^{2ij} \equiv \sigma^{ij}_{\dot\alpha_1\dot\beta_1}Z^{2\dot\alpha_1\dot\beta_1}\,, \quad \big(Y^2Z^2\big)^{ij} \equiv Y^{2k(i}Z^{2j)k} \tag{D.8}$$

and analogously for the primed indices. We have also introduced the following quantities quadratic and cubic in Y and symmetric in spinor indices

$$Y^2_{\alpha_1\beta_1} \equiv Y_{\alpha_1\alpha_2}Y^{\alpha_2}_{\beta_1}\,, \qquad Y^2_{\alpha_2\beta_2} \equiv Y_{\alpha_1\alpha_2}Y^{\alpha_1}_{\beta_2}\,, \tag{D.9}$$

$$Y^3_{\alpha_1\beta_2} \equiv Y^2_{\alpha_1\beta_1}Y^{\beta_1}_{\beta_2} = -Y^2_{\beta_2\alpha_2}Y^{\alpha_2}_{\alpha_1}\,, \tag{D.10}$$

and quartic in Y and antisymmetric in spinor indices

$$Y^4_{\alpha_1\beta_1} \equiv Y^2_{\alpha_1\gamma_1}Y^{2\gamma_1}{}_{\beta_1} = -\frac{1}{2}\epsilon_{\alpha_1\beta_1}Y^4\,, \qquad Y^4_{\alpha_2\beta_2} \equiv Y^2_{\alpha_2\gamma_2}Y^{2\gamma_2}{}_{\beta_2} = \frac{1}{2}\epsilon_{\alpha_2\beta_2}Y^4\,, \tag{D.11}$$

where

$$Y^4 \equiv Y^2_{\alpha_1\beta_1}Y^{2\alpha_1\beta_1} = -Y^2_{\alpha_2\beta_2}Y^{2\alpha_2\beta_2}\,. \tag{D.12}$$

The spinorial quantities s and t are defined as

$$s(Y) \equiv Y + \frac{i}{3}Y^3\,, \qquad t(Y) \equiv \epsilon + iY^2 - \frac{1}{6}Y^4\,. \tag{D.13}$$

Analogous definitions can be given for Z. The normalization of the dynamical generators is not fixed by the superalgebra at order $\mathcal{O}(g_2)$ and can be an arbitrary (dimensionless) function $f(\mu\alpha_3\,, \frac{\alpha_1}{\alpha_3})$ of the light-cone momenta and μ due to the fact that P^+ is a central element of the algebra.

D.2 Commutation relations

Rules for (anti)commutation of β annihilation operator with Y and Z elements in the prefactor:

$$Y^{\alpha_1\alpha_2} = \sum_{r=1}^{3}\sum_{n} G_{|n|\,(r)}\beta^{\dagger\alpha_1\alpha_2}_{n\,(r)} \tag{D.14}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^{\alpha_1\,\alpha_2}\} = G_{|m|\,(s)}\delta^{\alpha_1}_{\gamma_1}\delta^{\alpha_2}_{\gamma_2} \tag{D.15}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y_{\alpha_1\,\alpha_2}\} = G_{|m|\,(s)}\epsilon_{\alpha_1\,\gamma_1}\epsilon_{\alpha_2\,\gamma_2} \tag{D.16}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^2_{\alpha_1\,\beta_1}\} = G_{|m|\,(s)}\left(\epsilon_{\gamma_1\,\alpha_1}Y_{\beta_1\,\gamma_2} + \epsilon_{\gamma_1\,\beta_1}Y_{\alpha_1\,\gamma_2}\right) \tag{D.17}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^2_{\alpha_2\,\beta_2}\} = G_{|m|\,(s)}\left(\epsilon_{\gamma_2\,\alpha_2}Y_{\gamma_1\,\beta_2} + \epsilon_{\gamma_2\,\beta_2}Y_{\gamma_1\,\alpha_2}\right) \tag{D.18}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^3_{\alpha_1\,\beta_2}\} = G_{|m|\,(s)}\left(\epsilon_{\gamma_1\,\alpha_1}Y^2_{\gamma_2\,\beta_2} - \epsilon_{\gamma_2\,\beta_2}Y^2_{\alpha_1\,\gamma_1} + Y_{\alpha_1\,\gamma_2}\,Y_{\gamma_1\,\beta_2}\right) \tag{D.19}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^4\} = -4\,G_{|m|\,(s)}Y^3_{\gamma_1\,\gamma_2} \tag{D.20}$$

And exactly the same for Z and the dotted indices

D.3 Matrix elements

Some useful matrix elements are (where $\langle 3| \equiv \langle\alpha_3|$)

$$\langle 3|\,\alpha^{(3)\,i}_n\,\widetilde{K}^{\dot\gamma_1\gamma_1}|V\rangle = \left(K^{(3)}_{-n}\,\sigma^{i\dot\gamma_1\gamma_1} + \widetilde{K}^{\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n\,p}\,\alpha^{\dagger\,(s)\,i}_p\right)\langle 3|V\rangle \tag{D.21}$$

Where s and any other internal string index is restricted to run over $1,2$ only. We also have,

$$\langle 3|\,\alpha^{(3)\,i}_n\,K_k\widetilde{K}_l|V\rangle = \left(K^{(3)}_n\,\widetilde{K}_l\,\delta^{ik} + K^{(3)}_{-n}\,K_k\,\delta^{il} + K_k\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n\,p}\,\alpha^{\dagger\,(s)\,i}_p\right)\langle 3|V\rangle \tag{D.22}$$

$$\begin{aligned}\langle 3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,K_k\,\widetilde{K}_l|V\rangle = \Bigg(& K^{(3)}_{n_1}\,K^{(3)}_{-n_2}\,\delta^{ik}\,\delta^{jl} + K^{(3)}_{-n_1}\,K^{(3)}_{n_2}\,\delta^{il}\,\delta^{jk}\\ &+ K^{(3)}_{n_1}\,\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p\,\delta^{ik} + K^{(3)}_{n_2}\,\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\,\delta^{jk}\\ &+ K^{(3)}_{-n_1}\,K_k\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p\,\delta^{il} + K^{(3)}_{-n_2}\,K_k\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\,\delta^{jl}\\ &+ K_k\,\widetilde{K}_l\left(\widetilde{N}^{3\,3}_{n_1\,n_2}\delta^{ij} + \widetilde{N}^{3\,s}_{n_1\,p}\,\widetilde{N}^{3\,r}_{n_2\,q}\,\alpha^{\dagger\,(s)\,i}_p\,\alpha^{\dagger\,(r)\,j}_q\right)\Bigg)\langle 3|V\rangle\end{aligned} \tag{D.23}$$

$$\langle 3|\,\beta^{(3)}_{n\,\sigma_1\sigma_2}\,t_{\beta_1\gamma_1}(Y)\,|V\rangle = \Bigg\{\left(\widetilde{Q}^{3\,s}_{n\,p} - \widetilde{Q}^{s\,3}_{p\,n}\right)\,\beta^{\dagger\,(s)}_{p\,\sigma_1\sigma_2}\left(\epsilon_{\beta_1\gamma_1} + iY^2_{\beta_1\gamma_1} + \frac{1}{12}\,\epsilon_{\beta_1\gamma_1}Y^4\right)$$
$$+\,iG^{(3)}_{|n|}\,(\epsilon_{\sigma_1\beta_1}\,Y_{\gamma_1\sigma_2} + \epsilon_{\sigma_1\gamma_1}\,Y_{\beta_1\sigma_2}) - \frac{1}{3}G^{(3)}_{|n|}\,\epsilon_{\beta_1\gamma_1}\,Y^3_{\sigma_1\sigma_2}\Bigg\}\langle 3|V\rangle \tag{D.24}$$

$$\langle 3|\,\beta^{(3)}_{n\,\sigma_1\sigma_2}\,t_{\beta_2\gamma_2}(Y)\,|V\rangle = \Bigg\{\left(\widetilde{Q}^{3\,s}_{n\,p} - \widetilde{Q}^{s\,3}_{p\,n}\right)\,\beta^{\dagger\,(s)}_{p\,\sigma_1\sigma_2}\left(\epsilon_{\beta_2\gamma_2} + iY^2_{\beta_2\gamma_2} - \frac{1}{12}\,\epsilon_{\beta_2\gamma_2}Y^4\right)$$
$$+\,iG^{(3)}_{|n|}\,(\epsilon_{\sigma_2\beta_2}\,Y_{\sigma_1\gamma_2} + \epsilon_{\sigma_2\gamma_2}\,Y_{\sigma_1\beta_2}) + \frac{1}{3}G^{(3)}_{|n|}\,\epsilon_{\beta_2\gamma_2}\,Y^3_{\sigma_1\sigma_2}\Bigg\}\langle 3|V\rangle \tag{D.25}$$

$$\langle 3|\,\beta^{(3)}_{n\,\sigma_1\sigma_2}\,s_{\beta_1\gamma_2}(Y)\,|V\rangle = \Bigg\{\left(\widetilde{Q}^{3\,s}_{n\,p} - \widetilde{Q}^{s\,3}_{p\,n}\right)\,\beta^{\dagger\,(s)}_{p\,\sigma_1\sigma_2}\left(Y_{\beta_1\gamma_2} + \frac{i}{3}Y^3_{\beta_1\gamma_2}\right) + G^{(3)}_{|n|}\,\epsilon_{\sigma_1\beta_1}\,\epsilon_{\sigma_2\gamma_2}$$
$$+\,\frac{i}{3}G^{(3)}_{|n|}\left(\epsilon_{\sigma_1\beta_1}\,Y^2_{\sigma_2\gamma_2} - \epsilon_{\sigma_2\beta_2}\,Y^2_{\sigma_1\beta_1} + Y_{\beta_1\sigma_2}\,Y_{\sigma_1\gamma_2}\right)\Bigg\}\langle 3|V\rangle \tag{D.26}$$

$$\langle\alpha_3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,Q_{2\;\alpha_1\,\dot\alpha_2} = \frac{\bar\eta}{\sqrt{2\,|\alpha_3|}}\langle\alpha_3|\,\left(\Omega_{n_2}\,\sigma^{j\,\dot\beta_1}_{\;\;\alpha_1}\,\alpha^i_{n_1}\,\beta_{n_2\;\dot\beta_1\dot\alpha_2} + \quad (i\leftrightarrow j,\,n_1\leftrightarrow n_2)\right) \tag{D.27}$$

We are now prepared to construct the matrix elements we need certain calculations, for instance,

$$\epsilon^{\alpha_1\beta_1}\,\epsilon^{\dot\alpha_2\dot\beta_2}\,\langle 3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,Q_{2\;\alpha_1\,\dot\alpha_2}\,|Q_{3\;\beta_1\,\dot\beta_2}\rangle = \frac{g_2}{4\,\alpha_3^3}\,\sqrt{\frac{-\alpha'\,\kappa}{-4\,\alpha_3}}\,\Omega_{n_2}\,\sigma^{j\,\dot\gamma_1}_{\;\;\alpha_1}\epsilon^{\alpha_1\beta_1}\,\epsilon^{\dot\alpha_2\dot\beta_2}$$
$$\times\Bigg[\left\{K^{(3)}_{-n_1}\,\sigma^{i\dot\sigma_1\sigma_1} + \widetilde{K}^{\dot\sigma_1\sigma_1}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\right\}\,t_{\beta_1\sigma_1}(Y)\,\Bigg\{\left(Z_{\dot\sigma_1\dot\beta_2} + \frac{i}{3}Z^3_{\dot\sigma_1\dot\beta_2}\right)\,\widehat{Q}^{3\,r}_{n_2\,q}\,\beta^{\dagger\,(r)}_{q\;\dot\gamma_1\dot\alpha_2}$$
$$+\,G^{(3)}_{|n_2|}\,\epsilon_{\dot\gamma_1\dot\sigma_1}\,\epsilon_{\dot\alpha_2\dot\beta_2} + \frac{i}{3}G^{(3)}_{|n_2|}\left(\epsilon_{\dot\gamma_1\dot\sigma_1}\,Z^2_{\dot\alpha_2\dot\beta_2} - \epsilon_{\dot\alpha_2\dot\beta_2}\,Z^2_{\dot\gamma_1\dot\sigma_1} + Z_{\dot\sigma_1\dot\alpha_2}\,Z_{\dot\gamma_1\dot\beta_2}\right)\Bigg\}$$
$$+\,i\,\widetilde{K}^{\dot\sigma_2\sigma_2}\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\,s_{\beta_1\sigma_2}(Y)\Bigg\{\widehat{Q}^{3\,r}_{n_2\,q}\,\beta^{\dagger\,(r)}_{q\;\dot\gamma_1\dot\alpha_2}\left(\epsilon_{\dot\beta_2\dot\sigma_2} - iZ^2_{\dot\beta_2\dot\sigma_2} - \frac{1}{12}\,\epsilon_{\dot\beta_2\dot\sigma_2}\,Z^4\right) - \frac{1}{3}\,G^{(3)}_{|n_2|}\,\epsilon_{\dot\beta_2\dot\sigma_2}\,Z^3_{\dot\gamma_1\dot\alpha_2}$$
$$+\;i\,G^{(3)}_{|n_2|}\,\left(\epsilon_{\dot\alpha_2\dot\beta_2}\,Z_{\dot\gamma_1\dot\sigma_2} + \epsilon_{\dot\alpha_2\dot\sigma_2}\,Z_{\dot\gamma_1\dot\beta_2}\right)\Bigg\}\Bigg]\langle 3|V\rangle \quad + \quad (i\leftrightarrow j, n_1\leftrightarrow n_2) \tag{D.28}$$

where $\widehat{Q} = \widetilde{Q} - \widetilde{Q}^T$.

$$
\begin{aligned}
\langle 3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,|H_3\rangle = \frac{g_2\,\alpha'}{8\,\alpha_3^3}\Bigg[&\Big(\widetilde{N}^{3\,3}_{n_1\,n_2}\,\delta^{ij} + \widetilde{N}^{3\,s}_{n_1\,p}\,\widetilde{N}^{3\,r}_{n_2\,q}\,\alpha^{\dagger\,(s)\,i}_p\,\alpha^{\dagger\,(r)\,j}_q\Big)\\
&\times\Bigg(\Big[K_k\widetilde{K}_l - \frac{\mu\,\alpha}{\alpha'}\,\delta_{kl}\Big]\;v^{kl} - \Big[K_{k'}\widetilde{K}_{l'} - \frac{\mu\,\alpha}{\alpha'}\,\delta_{k'l'}\Big]\;v^{k'l'}\\
&- K^{\dot\rho_1\rho_1}\,\widetilde{K}^{\dot\rho_2\rho_2}\,s_{\rho_1\rho_2}(Y)\,s^*_{\dot\rho_1\dot\rho_2}(Z) - \widetilde{K}^{\dot\rho_1\rho_1}\,K^{\dot\rho_2\rho_2}\,s^*_{\rho_1\rho_2}(Y)\,s_{\dot\rho_1\dot\rho_2}(Z)\Bigg)\\
&+\Bigg(K^{(3)}_{n_1}\,K^{(3)}_{-n_2}\,\delta^{ik}\,\delta^{jl} + K^{(3)}_{-n_1}\,K^{(3)}_{n_2}\,\delta^{il}\,\delta^{jk} + K^{(3)}_{n_1}\,\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p\,\delta^{ik} + K^{(3)}_{n_2}\,\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\,\delta^{jk}\\
&+ K^{(3)}_{-n_1}\,K_k\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p\,\delta^{il} + K^{(3)}_{-n_2}\,K_k\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\,\delta^{jl}\Bigg)\,v^{kl}\\
&-\Big(\sigma^{i\dot\rho_1\rho_1}\,K^{(3)}_{n_1}\,\widetilde{K}^{\dot\rho_2\rho_2}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p + \sigma^{j\dot\rho_1\rho_1}\,K^{(3)}_{n_2}\,\widetilde{K}^{\dot\rho_2\rho_2}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\Big)\,s_{\rho_1\rho_2}(Y)\,s^*_{\dot\rho_1\dot\rho_2}(Z)\\
&-\Big(\sigma^{i\dot\rho_1\rho_1}\,K^{(3)}_{-n_1}\,K^{\dot\rho_2\rho_2}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p + \sigma^{j\dot\rho_1\rho_1}\,K^{(3)}_{-n_2}\,K^{\dot\rho_2\rho_2}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\Big)\,s^*_{\rho_1\rho_2}(Y)\,s_{\dot\rho_1\dot\rho_2}(Z)\Bigg]\langle 3|V\rangle
\end{aligned}
\tag{D.29}
$$

D.4 More matrix elements

Consider

$$
\begin{aligned}
\langle 3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,\widetilde{K}^{\dot\gamma_1\gamma_1}|V\rangle = (K^{(3)}_{-n_1}\,\sigma^{i\,\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p + K^{(3)}_{-n_2}\,\sigma^{j\,\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\\
+\widetilde{K}^{\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,3}_{n_1\,n_2}\,\delta^{i\,j})\langle 3|V\rangle
\end{aligned}
\tag{D.30}
$$

and therefore

$$
\begin{aligned}
\langle 3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,|Q_{3\,\beta_1\dot\beta_2}\rangle &= \frac{g_2\eta}{4\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\\
\times\Big\{s_{\dot\gamma_1\dot\beta_2}(Z)t_{\beta_1\gamma_1}(Y)\Big[&K^{(3)}_{-n_1}\,\sigma^{i\,\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p + K^{(3)}_{-n_2}\,\sigma^{j\,\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\\
&+\widetilde{K}^{\dot\gamma_1\gamma_1}\,\Big(\widetilde{N}^{3\,3}_{n_1\,n_2}\,\delta^{i\,j} + \widetilde{N}^{3\,r}_{n_1\,q}\,\alpha^{\dagger\,(r)\,i}_q\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p\Big)\Big]\\
+\,is_{\beta_1\gamma_2}(Y)t^*_{\dot\beta_2\dot\gamma_2}(Z)\widetilde{K}^{\dot\gamma_2\gamma_2}\,&\Big(\widetilde{N}^{3\,3}_{n_1\,n_2}\,\delta^{i\,j} + \widetilde{N}^{3\,r}_{n_1\,q}\,\alpha^{\dagger\,(r)\,i}_q\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p\Big)\Big\}
\end{aligned}
\tag{D.31}
$$

We will need the following expressions:

$$
\begin{aligned}
w^{i\,j}_{n\,\dot\beta_1\dot\alpha_2} \equiv [\beta_{n\,\dot\beta_1\dot\alpha_2}, v^{i\,j}] = G^{(3)}_{|n|}\Bigg\{&\delta^{i\,j}\left(-\frac{1}{3}Z^3_{\dot\beta_1\dot\alpha_2} - \frac{1}{36}Y^4 Z^3_{\dot\beta_1\dot\alpha_2}\right)\\
&+\frac{i}{2}\left[\frac{1}{3}Y^{2\,i\,j}Z^3_{\dot\beta_1\dot\alpha_2} - \sigma^{i\,j}_{\dot\rho_1\dot\gamma_1}\left(\delta^{\dot\rho_1}_{\dot\beta_1}Z^{\dot\gamma_1}_{\dot\alpha_2} + \delta^{\dot\gamma_1}_{\dot\beta_1}Z^{\dot\rho_1}_{\dot\alpha_2}\right)\left(1+\frac{1}{12}Y^4\right)\right] - \frac{1}{2}Y^{2\,k\,(i}\sigma^{j)\,k}_{\dot\beta_1\dot\gamma_1}\,Z^{\dot\gamma_1}_{\dot\alpha_2}\Bigg\}
\end{aligned}
\tag{D.32}
$$

$$w^{i'\,j'}_{n\,\dot\beta_1\dot\alpha_2} \equiv [\beta_{n\,\dot\beta_1\dot\alpha_2}, v^{i'\,j'}] = G^{(3)}_{|n|}\Bigg\{\delta^{i'\,j'}\left(\frac{1}{3}Z^3_{\dot\beta_1\dot\alpha_2} - \frac{1}{36}Y^4 Z^3_{\dot\beta_1\dot\alpha_2}\right)$$
$$+\frac{i}{2}\left[-\frac{1}{3}Y^{2\,i'\,j'}Z^3_{\dot\beta_1\dot\alpha_2} - \sigma^{i'\,j'}_{\dot\rho_2\dot\gamma_2}\left(\delta^{\dot\rho_2}_{\dot\alpha_2}Z^{\dot\gamma_2}_{\dot\beta_1} + \delta^{\dot\gamma_2}_{\dot\alpha_2}Z^{\dot\rho_2}_{\dot\beta_1}\right)\left(1-\frac{1}{12}Y^4\right)\right] - \frac{1}{2}Y^{2\,k'\,(i'}\sigma^{j')\,k'}_{\dot\alpha_2\dot\gamma_2}Z^{\dot\gamma_2}_{\dot\beta_1}\Bigg\} \tag{D.33}$$

We'll also need the following

$$Q_{2\;\alpha_1\,\dot\alpha_2}\,\alpha^{\dagger\,(3)\,k}_{n_1}\,\alpha^{\dagger\,(3)\,l}_{n_2}\,|\alpha_3\rangle = \frac{-\eta}{\sqrt{2\,|\alpha_3|}}\,\Big(\Omega_{n_1}\,\sigma^{k\,\dot\gamma_1}{}_{\alpha_1}\,\alpha^{\dagger\,l}_{n_2}\,\beta^{\dagger}_{n_1\;\dot\gamma_1\dot\alpha_2} + \quad (k\leftrightarrow l,\, n_1\leftrightarrow n_2)\Big)\,|\alpha_3\rangle \tag{D.34}$$

or

$$|\lambda\rangle = Q_2^{\beta_1\,\dot\beta_2}\,\alpha^{\dagger\,(3)\,k}_{n_1}\,\alpha^{\dagger\,(3)\,l}_{n_2}\,|\alpha_3\rangle = \frac{-\eta}{\sqrt{2\,|\alpha_3|}}\,\Big(\Omega_{n_1}\,\sigma^{k\,\dot\gamma_1\,\beta_1}\,\alpha^{\dagger\,l}_{n_2}\,\beta^{\dagger\,\dot\beta_2}_{n_1\;\dot\gamma_1} + \quad (k\leftrightarrow l,\, n_1\leftrightarrow n_2)\Big)\,|\alpha_3\rangle \tag{D.35}$$

allowing us to calculate,

$$\langle\lambda|H_3\rangle = \frac{g_2\,\alpha'}{8\,\alpha_3^3}\frac{\bar\eta}{\sqrt{2|\alpha_3|}}\sigma^{k}{}_{\beta_1\,\dot\gamma_1}\,\Omega_{n_1}$$
$$\times\Bigg[\left(-\beta^{\dagger\,\dot\gamma_1}_{q\,(r)\,\dot\beta_2}\,\widehat{Q}^{3\,r}_{n_1\,q}\right)\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,l}_{p}\Bigg\{\left[K_i\widetilde{K}_j - \frac{\mu\,\alpha}{\alpha'}\,\delta_{ij}\right]\,v^{ij} - \left[K_{i'}\widetilde{K}_{j'} - \frac{\mu\,\alpha}{\alpha'}\,\delta_{i'j'}\right]\,v^{i'j'}$$
$$- K^{\dot\rho_1\rho_1}\,\widetilde{K}^{\dot\rho_2\rho_2}\,s_{\rho_1\rho_2}(Y)\,s^*_{\dot\rho_1\dot\rho_2}(Z) - \widetilde{K}^{\dot\rho_1\rho_1}\,K^{\dot\rho_2\rho_2}\,s^*_{\rho_1\rho_2}(Y)\,s_{\dot\rho_1\dot\rho_2}(Z)\Bigg\}$$
$$+\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,l}_{p}\Bigg\{\left[K_i\widetilde{K}_j - \frac{\mu\,\alpha}{\alpha'}\,\delta_{ij}\right]\,(w^{ij}_{n_1})^{\dot\gamma_1}_{\dot\beta_2} - \left[K_{i'}\widetilde{K}_{j'} - \frac{\mu\,\alpha}{\alpha'}\,\delta_{i'j'}\right]\,\left(w^{i'j'}_{n_1}\right)^{\dot\gamma_1}_{\dot\beta_2}$$
$$+G^{(3)}_{|n_1|}\,K^{\dot\rho_1\rho_1}\,\widetilde{K}^{\dot\rho_2\rho_2}\,s_{\rho_1\rho_2}(Y)\left[\delta^{\dot\gamma_1}_{\dot\rho_1}\,\epsilon_{\dot\beta_2\,\dot\rho_2} - \frac{i}{3}\left(\delta^{\dot\gamma_1}_{\dot\rho_1}\,Z^2_{\dot\beta_2\,\dot\rho_2} - \epsilon_{\dot\beta_2\,\dot\rho_2}\,Z^{2\,\dot\gamma_1}_{\dot\rho_1} + Z_{\dot\rho_1\,\dot\beta_2}\,Z^{\dot\gamma_1}_{\dot\rho_2}\right)\right]$$
$$+G^{(3)}_{|n_1|}\,\widetilde{K}^{\dot\rho_1\rho_1}\,K^{\dot\rho_2\rho_2}\,s^*_{\rho_1\rho_2}(Y)\left[\delta^{\dot\gamma_1}_{\dot\rho_1}\,\epsilon_{\dot\beta_2\,\dot\rho_2} + \frac{i}{3}\left(\delta^{\dot\gamma_1}_{\dot\rho_1}\,Z^2_{\dot\beta_2\,\dot\rho_2} - \epsilon_{\dot\beta_2\,\dot\rho_2}\,Z^{2\,\dot\gamma_1}_{\dot\rho_1} + Z_{\dot\rho_1\,\dot\beta_2}\,Z^{\dot\gamma_1}_{\dot\rho_2}\right)\right]\Bigg\}$$
$$+\left(K^{(3)}_{n_2}\,\widetilde{K}_j\,\delta^{li} + K^{(3)}_{-n_2}\,K_i\,\delta^{lj}\right)\left\{\left(-\beta^{\dagger\,\dot\gamma_1}_{q\,(r)\,\dot\beta_2}\,\widehat{Q}^{3\,r}_{n_1\,q}\right)\,v^{ij} + (w^{ij}_{n_1})^{\dot\gamma_1}_{\dot\beta_2}\right\}$$
$$-\sigma^{l\,\dot\rho_1\,\rho_1}\,K^{(3)}_{n_2}\,\widetilde{K}^{\dot\rho_2\rho_2}\,s_{\rho_1\rho_2}(Y)\Bigg\{s^*_{\dot\rho_1\dot\rho_2}(Z)\,\left(-\beta^{\dagger\,\dot\gamma_1}_{q\,(r)\,\dot\beta_2}\,\widehat{Q}^{3\,r}_{n_1\,q}\right)$$
$$-G^{(3)}_{|n_1|}\,\left[\delta^{\dot\gamma_1}_{\dot\rho_1}\,\epsilon_{\dot\beta_2\,\dot\rho_2} - \frac{i}{3}\left(\delta^{\dot\gamma_1}_{\dot\rho_1}\,Z^2_{\dot\beta_2\,\dot\rho_2} - \epsilon_{\dot\beta_2\,\dot\rho_2}\,Z^{2\,\dot\gamma_1}_{\dot\rho_1} + Z_{\dot\rho_1\,\dot\beta_2}\,Z^{\dot\gamma_1}_{\dot\rho_2}\right)\right]\Bigg\}$$
$$-\sigma^{l\,\dot\rho_1\,\rho_1}\,K^{(3)}_{-n_2}\,K^{\dot\rho_2\rho_2}\,s^*_{\rho_1\rho_2}(Y)\Bigg\{s_{\dot\rho_1\dot\rho_2}(Z)\,\left(-\beta^{\dagger\,\dot\gamma_1}_{q\,(r)\,\dot\beta_2}\,\widehat{Q}^{3\,r}_{n_1\,q}\right)$$
$$-G^{(3)}_{|n_1|}\,\left[\delta^{\dot\gamma_1}_{\dot\rho_1}\,\epsilon_{\dot\beta_2\,\dot\rho_2} + \frac{i}{3}\left(\delta^{\dot\gamma_1}_{\dot\rho_1}\,Z^2_{\dot\beta_2\,\dot\rho_2} - \epsilon_{\dot\beta_2\,\dot\rho_2}\,Z^{2\,\dot\gamma_1}_{\dot\rho_1} + Z_{\dot\rho_1\,\dot\beta_2}\,Z^{\dot\gamma_1}_{\dot\rho_2}\right)\right]\Bigg\}\Bigg]$$
$$+\quad (k\leftrightarrow l,\, n_1\leftrightarrow n_2) \tag{D.36}$$

D.5 Example calculation

There is a more direct method of calculating amplitudes without resorting to the intermediate state projectors introduced in section 2.2.5. Consider a 2-string $\to$ 2-string operator M; as an example M could be $\langle e|H_3\rangle\langle H_3|e\rangle$, where $|e\rangle$ is some external string state. Consider the spacetime index structure of the matrix element

$$\langle \alpha^i_{n_1}\alpha^j_{n_2}|M|\alpha^{k\,\dagger}_{n_1}\alpha^{l\,\dagger}_{n_2}\rangle = A\,\delta^{ij}\delta^{kl} + B\,\delta^{ik}\delta^{jl} + C\,\delta^{il}\delta^{jk}. \tag{D.37}$$

Based on this structure, we now consider the $|[\mathbf{9},\mathbf{1}]\rangle$ state

$$|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} = \frac{1}{\sqrt{2}}\left(\alpha^{\dagger\,i}_n\alpha^{\dagger\,j}_{-n} + \alpha^{\dagger\,j}_n\alpha^{\dagger\,i}_{-n} - \frac{1}{2}\delta^{ij}\alpha^{\dagger\,k}_n\alpha^{\dagger\,k}_{-n}\right)|\alpha\rangle. \tag{D.38}$$

We find that

$$^{(i,j)}\langle[\mathbf{9},\mathbf{1}]|M|[\mathbf{9},\mathbf{1}]\rangle^{(i,j)} = (B+C)\left(1+\frac{1}{2}\delta^{ij}\right) \tag{D.39}$$

where $1+\frac{1}{2}\delta^{ij}$ is the normalization of the $|[\mathbf{9},\mathbf{1}]\rangle$ state, and is understood to be dropped in calculating an energy shift. Thus in calculating such a shift, we are instructed to simply calculate B and C in (D.37).

The method is to take $|e\rangle = \alpha^{\dagger\,k}_{n_1}\alpha^{\dagger\,l}_{n_2}|\alpha_3\rangle$, and so for a general amplitude involving the 3-string states $|A\rangle$ and $|B\rangle$, we expand the 2-string states $\langle e|A\rangle$ and $\langle e|B\rangle$ to the desired order in intermediate oscillators and calculate

$$\langle A|\alpha^{\dagger\,k}_{n_1}\alpha^{\dagger\,l}_{n_2}|\alpha_3\rangle\langle\alpha_3|\,\alpha^i_{n_1}\alpha^j_{n_2}|B\rangle \tag{D.40}$$

where for the $|[\mathbf{9},\mathbf{1}\rangle$ state we sum only those contributions proportional to either $\delta^{ik}\delta^{jl}$ or $\delta^{il}\delta^{jk}$. For example if the contact term were being calculated, $|A\rangle = |B\rangle = |Q_3\rangle$, and we would use (D.31) expanded to quadratic order in oscillators in order to capture the impurity conserving channel contribution. Of course the appropriate level matching must be enforced, and we further have that

$$\langle 3\text{ string vacuum}|\alpha_3\rangle\langle\alpha_3|3\text{ string vacuum}\rangle = 2r(1-r) \tag{D.41}$$

where the factor of 2 counts the two ways of contracting the internal string vacuua between right and left. Finally, the internal momenta must be integrated over via $\int_0^1 d\,r$.

As an example we calculate the contribution from the double fermionic intermediate state to the H_3 term of the mass shift for the trace state of section 2.3.1. We take those pieces of (D.29) quadratic in fermionic oscillators (see the fourth line of (D.29))

$$\begin{aligned}\langle\alpha_3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,|H_3\rangle = f\frac{g_2\,\alpha'}{8\,\alpha_3^3}&\left(K^{(3)}_{n_1}\,K^{(3)}_{-n_2}\,\delta^{ik}\,\delta^{jl} + K^{(3)}_{-n_1}\,K^{(3)}_{n_2}\,\delta^{il}\,\delta^{jk}\right)\delta^{kl}\\ &\times\left(\frac{1}{2}\widehat{Q}^{r_1\,r_2}_{q_1\,q_2}\big(\beta^{\alpha_1\alpha_2\,\dagger}_{q_1(r_1)}\beta^{\dagger}_{q_2(r_2)\,\alpha_1\alpha_2} - \beta^{\dot\alpha_1\dot\alpha_2\,\dagger}_{q_1(r_1)}\beta^{\dagger}_{q_2(r_2)\,\dot\alpha_1\dot\alpha_2}\big)\right)\\ &\times\langle\alpha_3|3\text{ string vacuum}\rangle\end{aligned} \tag{D.42}$$

where we have taken the leading delta function term of v^{kl} (D.8). Another contribution stemming from the quadratic pieces of v^{kl} will be zero here because for the trace state we take $i = j$ and sum. This kills the antisymmetric combination of σ-matrices found in the quadratic terms of v^{kl}. Taking $n_1 = n = -n_2$ and acting $\frac{1}{2}\delta^{ij}$ on the above element, and then taking it's square modulus, we find

$$\left|\frac{1}{2}\langle\alpha_3|\,\alpha_n^{(3)\,i}\,\alpha_{-n}^{(3)\,i}\,|H_3\rangle\right|^2 = |f|^2\left(\frac{g_2\alpha'}{4\alpha_3^3}\right)^2\left[K_n^2+K_{-n}^2\right]^2\left(\frac{1}{2}\widehat{Q}^{r_1\,r_2}_{q_1\,q_2}\right)^*\frac{1}{2}\widehat{Q}^{s_1\,s_2}_{p_1\,p_2}$$
$$\langle 3 \text{ string vacuum}|\alpha_3\rangle\left(\beta^{\alpha_1\alpha_2}_{q_2(r_2)}\beta_{q_1(r_1)\,\alpha_1\alpha_2} - \beta^{\dot\alpha_1\dot\alpha_2}_{q_2(r_2)}\beta_{q_1(r_1)\,\dot\alpha_1\dot\alpha_2}\right)$$
$$\left(\beta^{\dagger\,\beta_1\beta_2}_{p_1(s_1)}\beta^{\dagger}_{p_2(s_2)\,\beta_1\beta_2} - \beta^{\dagger\,\dot\beta_1\dot\beta_2}_{p_1(s_1)}\beta^{\dagger}_{p_2(s_2)\,\dot\beta_1\dot\beta_2}\right)\langle\alpha_3|3 \text{ string vacuum}\rangle. \tag{D.43}$$

Commuting the β oscillators though one another gives the following factor (times two since dotted and undotted oscillators are orthogonal)

$$4\left(\delta_{q_1p_1}\delta^{r_1s_1}\delta_{q_2p_2}\delta^{r_2s_2} - \delta_{q_1p_2}\delta^{r_1s_2}\delta_{q_2p_1}\delta^{r_2s_1}\right). \tag{D.44}$$

Sums over s_i and p_i set these variables to their r_i and q_i counterparts indicated by the delta functions. Level matching is then imposed by adding the following factor before summing over r_i and q_i

$$(\delta^{r_1\,r_2}\delta_{q_1+q_2} + (1-\delta^{r_1\,r_2})\delta_{q_1}\delta_{q_2}) \tag{D.45}$$

where the second term counts the contribution from the zero modes where each intermediate string is excited by a single oscillator. For the purpose of this example, we will ignore these as we are interested in demonstrating the convergence of the sum over the remaining mode number. We have

$$\delta E = \int_0^1 dr\, 2r(1-r)|f|^2\left(\frac{g_2\alpha'}{4\alpha_3^3}\right)^2\left[K_n^2+K_{-n}^2\right]^2\sum_q\sum_{s=1}^{2}\frac{-\alpha_3}{2\left(\omega_n - r^{-1}\omega_q\right)}2\cdot 8\cdot\frac{1}{4}\left|\widehat{Q}^{s\,s}_{q\,-q}\right|^2 \tag{D.46}$$

where we have included the energy denominator, used (D.41), and noted that the second term in (D.44) gives the same result as the first on account of the antisymmetry of $\widehat{Q}^{r\,s}_{p\,q}$. The sum over the string label s just gives a factor of 2, as there is a 1$\leftrightarrow$2 string symmetry running through all equations. Also note that $f = r^{-1}(1-r)^{-1}$. The convergence of the sum is evident from the form of $\widehat{Q}^{1\,1}_{q\,-q}$ (see appendix C)

$$\left|\widehat{Q}^{1\,1}_{q\,-q}\right|^2 = \left(\frac{1}{4\pi}\right)^2\frac{q^2}{\omega_q^4} \sim \frac{1}{q^2} \tag{D.47}$$

and so the sum in (D.46) is convergent.

Appendix E

Plane-wave matrix model 2-loop effective action

E.1 The theta diagram

The "theta" diagram is given by the combination of the two three-vertices for the scalars of the first kind. It is the middle diagram in figure 3.3.
From the action we get the vertex as:

$$\frac{i\mu}{3}\,\mathrm{Tr}[\epsilon_{\bar{a}\bar{b}\bar{c}}X^{\bar{a}}X^{\bar{b}}X^{\bar{c}}] \tag{E.1}$$

so we can write the diagram as:

$$-\frac{1}{2}\frac{\mu^2}{3^2}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}\epsilon_{\bar{a}\bar{b}\bar{c}}\epsilon_{\bar{d}\bar{e}\bar{f}}X^{\bar{a}}_{ab}(\tau)X^{\bar{b}}_{bc}(\tau)X^{\bar{c}}_{ca}(\tau)X^{\bar{d}}_{de}(\tau')X^{\bar{e}}_{ef}(\tau')X^{\bar{f}}_{fd}(\tau') \tag{E.2}$$

There is therefore three propagators between $X(\tau)$ and $X(\tau')$'s the $\bar{a},\bar{b},\bar{c}$ and $\bar{d},\bar{e},\bar{f}$ all range over 1,2,3 and the ϵ limits them to the totally symmetric and totally antisymmetric combinations. That makes up for 6 on each side. Furthermore the requirement that the diagram be planar makes sure that if the $\bar{a},\bar{b},\bar{c}$ combination is symmetric then $\bar{d},\bar{e},\bar{f}$ can not be anti-symmetric and vice versa. The planar contractions also introduce a sign, due to their mixed symmetry. There is therefore 6×3 combinations of the propagators with the summation over the indices:

$$\frac{\mu^2R^3}{8\omega_1^3}\sum_{abc}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau'[g(\tau-\tau')+g^*(\tau'-\tau)]_{ab}[..]_{bc}[..]_{ca} \tag{E.3}$$

where

$$\omega_1=\frac{\mu}{3} \tag{E.4}$$

The diagram can then be written as

$$\frac{\mu^2R^3}{8\omega_1^3}\sum_{abc}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau'\left[\frac{\phi_{ab}^{(\tau'-\tau-\beta)}}{1-\phi_{ab}^{-\beta}}+\frac{\phi_{ab}^{*-(\tau'-\tau)}}{1-\phi_{ab}^{*-\beta}}\right][..]_{bc}\,[..]_{ca}\,\theta(\tau'-\tau) \tag{E.5}$$

$$+\left[\frac{\phi_{ab}^{*-(\tau'-\tau-\beta)}}{1-\phi_{ab}^{*-\beta}}+\frac{\phi_{ab}^{(\tau'-\tau)}}{1-\phi_{ab}^{-\beta}}\right][..]_{bc}[..]_{ca}\theta(\tau-\tau')$$

where ϕ_{ab} is defined in section 3.3.3. The expression can also be written as:

$$\frac{\mu^2 R^3}{8\omega_1^3}\sum_{abc}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau' \left[\frac{\mathcal{A}_{ab}\phi_{ab}^{(\tau'-\tau)}+\mathcal{B}_{ab}\phi_{ab}^{*-(\tau'-\tau)}}{C_{ab}}\right][..]_{bc}[..]_{ca}\theta(\tau'-\tau) \tag{E.6}$$
$$+\left[\frac{\mathcal{A}_{ab}^{*}\phi_{ab}^{*-(\tau'-\tau)}+\mathcal{B}_{ab}^{*}\phi_{ab}^{(\tau'-\tau)}}{C_{ab}}\right][..]_{bc}[..]_{ca}\theta(\tau-\tau')$$

where again, quantities are defined in section 3.3.3. It is possible to interchange τ and τ' in the second line of the integral in order to get everything multiplied by the same Heaviside function. The second line is then just a complex conjugate of the first because C_{ab} is a real quantity. It is then possible to write the diagram as:

$$\frac{\mu^2 R^3}{4\omega_1^3}\sum_{abc}\frac{1}{C_{ab}C_{bc}C_{ca}}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \int_{\tau}^{\frac{\beta}{2}} d\tau' \mathrm{Re}[(\mathcal{A}_{ab}\mathcal{A}_{bc}\mathcal{A}_{ca})(\phi_{ab}\phi_{bc}\phi_{ca})^{(\tau'-\tau)} \tag{E.7}$$
$$+(\mathcal{A}_{ab}\mathcal{A}_{bc}\mathcal{B}_{ca})(\phi_{ab}\phi_{bc}\phi_{ca}^{*-1})^{(\tau'-\tau)}+...+(\mathcal{B}_{ab}\mathcal{B}_{bc}\mathcal{B}_{ca})(\phi_{ab}^{*-1}\phi_{bc}^{*-1}\phi_{ca}^{*-1})^{(\tau'-\tau)}]$$

This can be simplified by noticing that:

$$\phi_{ab}\phi_{bc}\phi_{ca}=e^{3\omega_1},\phi_{ab}\phi_{bc}\phi_{ca}^{*-1}=e^{\omega_1},... \tag{E.8}$$

Then diagram is given by:

$$\frac{\mu^2 R^3}{4\omega_1^3}\sum_{abc}\frac{1}{C_{ab}C_{bc}C_{ca}}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \int_{\tau}^{\frac{\beta}{2}} d\tau' \mathrm{Re}(\mathcal{A}_{ab}\mathcal{A}_{bc}\mathcal{A}_{ca})e^{3\omega_1(\tau'-\tau)}+ \tag{E.9}$$
$$\mathrm{Re}(\mathcal{A}_{ab}\mathcal{A}_{bc}\mathcal{B}_{ca}+...)e^{\omega_1(\tau'-\tau)}+\mathrm{Re}(\mathcal{A}_{ab}\mathcal{B}_{bc}\mathcal{B}_{ca}+...)e^{-\omega_1(\tau'-\tau)}+\mathrm{Re}(\mathcal{B}_{ab}\mathcal{B}_{bc}\mathcal{B}_{ca})e^{-3\omega_1(\tau'-\tau)}$$

Performing the integration:

$$\frac{\mu^2 R^3}{4\omega_1^3}\sum_{abc}\frac{1}{C_{ab}C_{bc}C_{ca}}\left[\mathrm{Re}(\mathcal{A}_{ab}\mathcal{A}_{bc}\mathcal{A}_{ca})\frac{1+3\beta\omega_1-e^{3\beta\omega_1}}{9\omega_1^2}+\right. \tag{E.10}$$
$$\mathrm{Re}(\mathcal{A}_{ab}\mathcal{A}_{bc}\mathcal{B}_{ca}+...)\frac{1+\beta\omega_1-e^{\beta\omega_1}}{\omega_1^2}+\mathrm{Re}(\mathcal{A}_{ab}\mathcal{B}_{bc}\mathcal{B}_{ca}+...)\frac{1-\beta\omega_1-e^{-\beta\omega_1}}{\omega_1^2}$$
$$\left.+\mathrm{Re}(\mathcal{B}_{ab}\mathcal{B}_{bc}\mathcal{B}_{ca})\frac{1-3\beta\omega_1-e^{-3\beta\omega_1}}{9\omega_1^2}\right]$$

Using the given definitions of $\mathcal{A}$ and $\mathcal{B}$ from (3.62) it is possible to simplify the above

$$\boxed{\begin{aligned}&\frac{27\beta R^3}{4\mu^2}\sum_{abc}\frac{1}{C_{ab}^{\omega_1}C_{bc}^{\omega_1}C_{ca}^{\omega_1}}\Big[1+e^{-2\beta\mu}-9\,e^{-4\beta\mu/3}+16\,e^{-\beta\mu}-9\,e^{-2\beta\mu/3}\\&\qquad+[\cos(\beta A_{ab})+\cos(\beta A_{bc})+\cos(\beta A_{ca})]\\&\qquad\qquad\times\left[2e^{-5\beta\mu/3}+2e^{-\beta\mu/3}-8e^{-4\beta\mu/3}-8e^{-2\beta\mu/3}+12e^{-\beta\mu}\right]\Big]\end{aligned}} \tag{E.11}$$

E.2 The figure-eight diagram

This section is dedicated to the calculation of the first diagram in figure 3.3.
This comes from expanding the action to first order, $\exp(-S) \simeq 1 - S$, and so we pick up a sign:

$$
\begin{aligned}
&\frac{R}{4}\int_{-\beta/2}^{\beta/2}\left\langle \mathrm{Tr}[X^i, X^j]^2\right\rangle d\tau = \frac{R}{2}\int_{-\beta/2}^{\beta/2}\sum_{i<j}\left\langle \mathrm{Tr}[X^i, X^j]^2\right\rangle d\tau \\
=&\frac{R}{2}\int_{-\beta/2}^{\beta/2}\sum_{a<b}\left\langle \mathrm{Tr}[X^{\bar{a}}, X^{\bar{b}}]^2\right\rangle d\tau + \frac{R}{2}\int_{-\beta/2}^{\beta/2}\sum_{i<j}\left\langle \mathrm{Tr}[X^i, X^j]^2\right\rangle d\tau \\
&+ \frac{R}{2}\int_{-\beta/2}^{\beta/2}\sum_{\bar{a}}\sum_{i}\left\langle \mathrm{Tr}[X^{\bar{a}}, X^i]^2\right\rangle d\tau
\end{aligned}
\tag{E.12}
$$

Let us consider one of the above terms

$$
\begin{aligned}
\mathrm{Tr}([X^{\bar{a}}, X^{\bar{b}}][X^{\bar{a}}, X^{\bar{b}}]) &= \mathrm{Tr}((X^{\bar{a}}X^{\bar{b}} - X^{\bar{b}}X^{\bar{a}})(X^{\bar{a}}X^{\bar{b}} - X^{\bar{b}}X^{\bar{a}})) \\
&= 2\,\mathrm{Tr}(X^{\bar{a}}X^{\bar{b}}X^{\bar{a}}X^{\bar{b}} - X^{\bar{a}}X^{\bar{a}}X^{\bar{b}}X^{\bar{b}})
\end{aligned}
\tag{E.13}
$$

While the first term has a non-planar contribution, hence the whole expression for the $\bar{a}$-flavor bosonic field would be

$$
\begin{aligned}
&R\sum_{\bar{a}<\bar{b}}\sum_{abcd}\int_{-\beta/2}^{\beta/2}(-\langle X^{\bar{a}}_{ab}X^{\bar{a}}_{bc}\rangle\langle X^{\bar{b}}_{cd}X^{\bar{b}}_{da}\rangle)\,d\tau \\
&\qquad = -R\sum_{\bar{a}<\bar{b}}\sum_{abcd}\int_{-\beta/2}^{\beta/2}\left(\frac{R}{2\omega_1}\right)^2\delta^{aa}\delta^{bb}\delta_{ac}\delta_{bb}\delta_{ac}\delta_{dd}P_{ab}(\omega_1)P_{cd}(\omega_1)
\end{aligned}
\tag{E.14}
$$

where

$$
P_{ab}(\omega) \equiv (g + g_-^*)^{\omega}_{ab}
\tag{E.15}
$$

as per (3.56) and (3.50). Therefore what we get is

$$
-\left(\frac{R^3}{4\omega_1^2}\right)\frac{3\cdot 2}{2}\sum_{abd}\int_{-\beta/2}^{\beta/2}P_{ab}(\omega_1)P_{ad}(\omega_1)\,d\tau
\tag{E.16}
$$

The only difference for the i-flavor would be the value of the $\Sigma_{i<j}$, hence putting everything together we find the following result for (E.12):

$$
\begin{aligned}
-\frac{R^3}{4}\sum_{abd}\int_{-\beta/2}^{\beta/2}\Bigg[&\frac{3\cdot 2}{2\omega_1^2}P_{ab}(\omega_1)P_{ad}(\omega_1) \\
&+ \frac{6\cdot 5}{2\omega_2^2}P_{ab}(\omega_2)P_{ad}(\omega_2) + \frac{6\cdot 3}{\omega_1\omega_2}P_{ab}(\omega_1)P_{ad}(\omega_2)\Bigg]d\tau
\end{aligned}
\tag{E.17}
$$

Considering that $\omega_1 = \mu/3$ and $\omega_2 = \mu/6$, then the final expression would be

$$-\frac{R^3}{4\mu^2}\sum_{abd}\int_{-\beta/2}^{\beta/2}\Big[27P_{ab}(\omega_1)P_{ad}(\omega_1) + 540P_{ab}(\omega_2)P_{ad}(\omega_2) + 324P_{ab}(\omega_1)P_{ad}(\omega_2)\Big]d\tau \quad \text{(E.18)}$$

Now, consider the product of two P's:

$$P_{ab}P_{ad} = (H_1\theta + G_1\bar{\theta})(H_2\theta + G_2\bar{\theta}) = H_1H_2\theta + G_1G_2\bar{\theta} \quad \text{(E.19)}$$

It so happens that, when τ is set equal to τ'

$$H_1 = H_2 = G_1 = G_2 = \frac{1}{C}(1 - e^{-2\beta\omega}) \quad \text{(E.20)}$$

Using the fact that $[\theta(s) + \theta(-s)]_{s=0} \equiv 1$, we arrive at the final form:

$$\boxed{-\frac{27\beta R^3}{4\mu^2}\sum_{abd}\left[\frac{[1-e^{-2\beta\mu/3}]^2}{C_{ab}^{\omega_1}C_{ad}^{\omega_1}} + 20\frac{[1-e^{-\beta\mu/3}]^2}{C_{ab}^{\omega_2}C_{ad}^{\omega_2}} + 12\frac{[1-e^{-2\beta\mu/3}][1-e^{-\beta\mu/3}]}{C_{ab}^{\omega_1}C_{ad}^{\omega_2}}\right]} \quad \text{(E.21)}$$

Appendix F

1/4 BPS Wilson loop - chiral primary correlator

F.1 Metric fluctuations

Given (4.53) and (4.51), we must construct the traceless symmetric double covariant derivative,

$$D_{(\mu}D_{\nu)} \equiv \frac{1}{2}\left(D_\mu D_\nu + D_\nu D_\mu\right) - \frac{1}{5} g_{\mu\nu}\, g^{\rho\sigma} D_{\rho\sigma}. \tag{F.1}$$

The action of $D_\mu D_\nu$ on a scalar field ϕ is,

$$D_\mu D_\nu \phi = \partial_\mu \partial_\nu \phi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \phi. \tag{F.2}$$

The Christoffel symbols for the AdS geometry (4.51) are,

$$\begin{array}{cccc} \Gamma^{r_i}_{\phi_i\phi_i} = -r_i & \Gamma^{y}_{\phi_i\phi_i} = \dfrac{r_i^2}{y} & \Gamma^{\phi_i}_{\phi_i r_i} = \dfrac{1}{r_i} & \Gamma^{\phi_i}_{\phi_i y} = -\dfrac{1}{y} \\ \Gamma^{y}_{r_i r_i} = \dfrac{1}{y} & \Gamma^{r_i}_{y r_i} = -\dfrac{1}{y} & \Gamma^{y}_{yy} = -\dfrac{1}{y} & \end{array} \tag{F.3}$$

where $i = 1, 2$. The trace of $D_\mu D_\nu\, \phi$ is given by,

$$g^{\mu\nu} D_\mu D_\nu = \sum_{i=1}^{2} \left(y^2 \partial_y^2 + y^2 \partial_{r_i}^2 + \frac{y^2}{r_i^2} \partial_{\phi_i}^2 - 3y\partial_y + \frac{y^2}{r_i} \partial_{r_i} \right) \phi \tag{F.4}$$

Because of (4.55), we only keep those terms of $D_{(\mu}D_{\nu)}$ which contain derivatives in y. These are,

$$D_{(y}D_{y)} = \frac{4}{5}\partial_y^2 + \frac{8}{5y}\partial_y, \qquad D_{(r_1}D_{r_1)} = \frac{1}{r_1^2} D_{(\phi_1}D_{\phi_1)} = -\frac{1}{5}\partial_y^2 - \frac{2}{5y}\partial_y. \tag{F.5}$$

We now note that since the derivatives will be acting on y^J from the propagator (4.55), we may replace $\partial_y^2 \to J(J-1)/y^2$ and $y^{-1}\partial_y \to J/y^2$. Therefore the metric fluctuations may be expressed as follows,

$$\begin{aligned} \delta g_{yy} &= \left[-\frac{6J}{5} + \frac{4}{J+1}\left(\frac{4}{5}J(J-1) + \frac{8}{5}J \right) \right] \frac{L^2}{y^2} = 2J\frac{L^2}{y^2} \\ \delta g_{r_1 r_1} &= \frac{1}{r_1^2}\delta g_{\phi_1\phi_1} = \left[-\frac{6J}{5} - \frac{4}{J+1}\left(\frac{1}{5}J(J-1) + \frac{2}{5}J \right) \right] \frac{L^2}{y^2} = -2J\frac{L^2}{y^2}. \end{aligned} \tag{F.6}$$

F.2 Spherical harmonics

The five-sphere is embedded in $\mathbb{R}^6$ in the following manner,

$$\begin{aligned} x^1 &= \sin\theta\cos\phi & x^2 &= \sin\theta\sin\phi \\ x^3 &= \cos\theta\sin\rho\cos\hat{\phi} & x^4 &= \cos\theta\sin\rho\sin\hat{\phi} \\ x^5 &= \cos\theta\cos\rho\cos\tilde{\phi} & x^6 &= \cos\theta\cos\rho\sin\tilde{\phi}, \end{aligned} \tag{F.7}$$

and has the metric

$$ds^2_{S^5} = d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, \left(d\rho^2 + \sin^2\rho\, d\hat{\phi}^2 + \cos^2\rho\, d\tilde{\phi}^2\right). \tag{F.8}$$

The embedding (4.52) takes $\rho = \pi/2, \hat{\phi} = 0$, or $x^4 = x^5 = x^6 = 0$. Note that $\rho \in [0, \pi/2]$ while $\theta \in [0, \pi]$. A general chiral primary normalized as in (4.47) may be written as,

$$\frac{2^{J/2}}{\sqrt{\lambda^J J}} C^{I_1 \dots I_J}\, \mathrm{Tr}\, \Phi_{I_1} \dots \Phi_{I_J} \tag{F.9}$$

where $C^{I_1 \dots I_J}$ is traceless symmetric and $C^{I_1 \dots I_J} C^{*I_1 \dots I_J} = 1$. The corresponding spherical harmonic is given by $Y_J(\theta, \phi) = C^{I_1 \dots I_J} x^{I_1} \dots x^{I_J}$. A properly normalized (i.e. (4.47)) operator built on $\mathrm{Tr}(u \cdot \Phi)^J$ will then correspond to

$$Y_J(\theta, \phi) = \mathcal{N}_J(u) \Big[u_1 \sin\theta\, \cos\phi + u_2 \sin\theta\, \sin\phi + u_3 \cos\theta \Big]^J \tag{F.10}$$

for some normalization $\mathcal{N}_J(u)$. If we choose $u_1 = u_2 = 0$ and $u_3 = \pm i u_4 = 1$, i.e. the operator $\mathrm{Tr}(\Phi_3 \pm i\Phi_4)^J / \sqrt{\lambda^J J}$, then $\mathcal{N}_J(u) = 2^{-J/2}$.

F.3 R-symmetry

Let $\mathcal{O}_J = \frac{1}{\sqrt{J\, \lambda^J}}\, \mathrm{Tr}\, (\Phi_1 + i\Phi_2)^J$, Let U be a rotation in the x^1-x^2 plane. Then

$$\langle \mathcal{O}_J(x)\, W[C_{1/4}] \rangle = \langle U\, \mathcal{O}_J(x)\, W[C_{1/4}]\, U^\dagger \rangle = \langle \mathcal{O}_J(U\, x)\, U\, W[C_{1/4}]\, U^\dagger \rangle \tag{F.11}$$

Examining $C_{1/4}$ in (4.44), we see that the spatial rotation acting on $W[C_{1/4}]$ may be realized by a shift in the contour parameter τ, which can in turn by compensated by an R-symmetry rotation R in the θ^1-θ^2 plane, $U\, W[C_{1/4}]\, U^\dagger = R\, W[C_{1/4}]\, R^\dagger$. Then,

$$\langle \mathcal{O}_J(x)\, W[C_{1/4}] \rangle = \langle R\, \mathcal{O}_J(Ux)\, R^\dagger\, W[C_{1/4}] \rangle. \tag{F.12}$$

The operator expansion coefficient depends on the leading asymptotic in large x which is a function of only the length of $C_{1/4}$ and x^2,

$$\langle \mathcal{O}_J(x)\, W[C_{1/4}] \rangle \simeq \left(\frac{2\pi R}{4\pi^2 x^2} \right)^J \xi_J + \dots \tag{F.13}$$

Performing the θ^1-θ^2 plane R-symmetry transformation on $\mathcal{O}_J$ multiplies it by a phase $\exp(iJ\phi)$ so that,

$$\langle R\,\mathcal{O}_J(Ux)\,R^\dagger\,W[C_{1/4}]\rangle \simeq e^{iJ\phi}\left(\frac{2\pi R}{4\pi^2(Ux)^2}\right)^J \xi_J + \ldots = e^{iJ\phi}\left(\frac{2\pi R}{4\pi^2 x^2}\right)^J \xi_J + \ldots \tag{F.14}$$

Using (F.12) and (F.13), we have $e^{iJ\phi}\,\xi_J = \xi_J$, i.e. $\xi_J = 0$.

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